

COINVARIANT ALGEBRAS AND FAKE DEGREES FOR SPIN WEYL GROUPS

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ABSTRACT. The coinvariant algebra of a Weyl group plays a fundamental role in several areas of mathematics. The fake degrees are the graded multiplicities of the irreducible modules of a Weyl group in its coinvariant algebra, and they were computed by Steinberg, Lusztig and Beynon-Lusztig. In this paper we formulate a notion of spin coinvariant algebra for every Weyl group and compute all the spin fake degrees, which are by definition the graded multiplicities of the simple modules of a spin Weyl group in the spin coinvariant algebra. The spin fake degrees are all shown to be palindromic polynomials.

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1. INTRODUCTION

1.1. Background. Let V be the irreducible reflection representation of a Weyl group W . The invariant algebra $(S^*V)^W$ and the coinvariant algebra $(S^*V)_W$ of W are fundamental objects which have connections and applications in many areas of geometry and representation theory. According to Chevalley, the coinvariant algebra $(S^*V)_W$ is a graded regular representation of W (see [Hu, Lu2]). Following Lusztig, the graded multiplicity of a simple W -character ρ in $(S^*V)_W$ is called the fake degree of ρ , and it is a polynomial in a variable t which specializes at $t = 1$ to the degree of ρ . The fake degrees were computed by Steinberg [Stn] in type A_n (where W is the symmetric group S_{n+1}), by Lusztig [Lu1] for type B_n and D_n , and by Beynon-Lusztig [BL] using computer calculations for the exceptional types. The formulation and computation of fake degrees have significant applications to finite groups of Lie type, which were systematically developed by Lusztig [Lu1, Lu2].

We start with a distinguished double cover \widetilde{W} for any Weyl group W :

$$(1.1) \quad 1 \longrightarrow \{1, z\} \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1.$$

Schur [Sch] in 1911 computed the Schur multiplier for the symmetric groups S_n and initiated the spin representation theory of S_n ; see [Joz2] for a clear new exposition based on a systematic use of superalgebras. The Schur multiplier of W has been computed by Ihara and Yokonuma [IY] (cf. [Kar]) to be \mathbb{Z}_2 or a product of multiple copies of \mathbb{Z}_2 . The double cover \widetilde{W} in (1.1) corresponds to the choice of 2-cocycle with all elements nontrivial in every copy of \mathbb{Z}_2 . Another description of \widetilde{W} is as follows. Assume that W is generated by s_1, \dots, s_n subject to the relations $(s_i s_j)^{m_{ij}} = 1$ for all i, j . The double cover \widetilde{W} is chosen so that the *spin Weyl group algebra* $\mathbb{C}W^- := \mathbb{C}\widetilde{W} / \langle z + 1 \rangle$ is then generated by t_1, \dots, t_n subject to the relations $(t_i t_j)^{m_{ij}} = (-1)^{1+m_{ij}}$. One notable feature of $\mathbb{C}W^-$ is that it is naturally a superalgebra with each t_i being odd.

1.2. Goal. The goal of this paper is to formulate and compute the *spin fake degrees* of all simple characters of $\mathbb{C}W^-$ (which are the graded multiplicities in so-called spin coinvariant algebras), for every Weyl group W ; except type A which was done and expressed in terms of shifted q -hook formula in [WW1, WW2]. The computation in this paper is carried out case-by-case. Neat closed q -hook formulas of spin fake degrees are obtained for W of classical types and the answers are listed explicitly in tables for the exceptional types.

1.3. Formulation. The first problem which we encounter is that no natural candidate for a graded regular representation of $\mathbb{C}W^-$ is immediately available. We get around the difficulty as follows.

The reflection representation V of W is naturally endowed with a W -invariant bilinear form (\cdot, \cdot) . The Clifford (super)algebra $\mathcal{C}l_V$ associated to $(V, (\cdot, \cdot))$ is acted upon by W as automorphisms, and the semi-direct product $\mathcal{C}l_V \rtimes W$ is naturally a superalgebra. Khongsap and the second author [KW1, Theorem 2.4] have established an isomorphism of superalgebras:

$$(1.2) \quad \Phi : \mathcal{C}l_V \rtimes W \xrightarrow{\simeq} \mathcal{C}l_V \otimes \mathbb{C}W^-.$$

In the case when W is a symmetric group, this was established by Sergeev [Se] and Yamaguchi [Ya]. Modules of a superalgebra A are always assumed to have a \mathbb{Z}_2 -graded structure compatible with the action of A unless specified otherwise. We shall denote by $|A|$ the underlying algebra of A .

The Clifford algebra $\mathcal{C}l_V$ is a simple superalgebra, and hence the isomorphism (1.2) induces a Morita super-equivalence between the superalgebras $\mathfrak{H}_W^\epsilon := \mathcal{C}l_V \rtimes W$ and $\mathbb{C}W^-$ (see Proposition 3.3), and the study of the representation theory of $\mathbb{C}W^-$ is essentially equivalent to the counterpart for \mathfrak{H}_W^ϵ . The tensor superalgebra

$$(1.3) \quad \mathcal{C}l_V \otimes (S^*V)_W$$

is naturally a graded regular representation of \mathfrak{H}_W^ϵ , and hence will be called the spin coinvariant algebra. (This goes back to Wan and the second author [WW1] for $W = S_n$.) The graded multiplicity of a simple \mathfrak{H}_W^ϵ -character χ in $\mathcal{C}l_V \otimes (S^*V)_W$ will be called the spin fake degree of χ and denoted by $P(\chi, t)$.

Under the Morita super-equivalence induced by Φ in (1.2), the simple \mathfrak{H}_W^c -module $\mathcal{C}l_V$ is shown to correspond to the basic spin $\mathbb{C}W^-$ -module \mathcal{B}_W (see Theorem 3.5), and $\mathcal{C}l_V \otimes (S^*V)_W$ corresponds to $\mathcal{B}_W \otimes (S^*V)_W$. Here the basic spin module \mathcal{B}_W is the pullback of the simple $\mathcal{C}l_V$ -module via a homomorphism $\mathbb{C}W^-$ to $\mathcal{C}l_V$ [Mo2], and the construction goes back to Schur for $W = S_n$ (cf. [Joz2]). The graded multiplicity of a simple $\mathbb{C}W^-$ -character χ^- in $\mathcal{B}_W \otimes (S^*V)_W$ is called the spin fake degree of χ^- and denoted by $P^-(\chi^-, t)$. It is shown that $P(\chi, t)$ and $P^-(\chi^-, t)$ essentially coincide, up to a possible factor of 2 which is determined by Proposition 3.8, when χ corresponds to χ^- under the super-equivalence.

1.4. Main results. To simplify notation, we will denote by X_n the Weyl group of type X_n and the associated spin group algebra as $\mathbb{C}X_n^-$ (except that in type A we write $\mathbb{C}S_n^-$); for example $\mathbb{C}B_n^-$ denotes the spin Weyl group algebra of type B_n .

For W of type B_n or D_n , the split classes of W (with respect to \widetilde{W}) were classified and the simple ungraded $|\mathbb{C}W^-|$ -modules were all constructed by Read [Re2]. For W of exceptional type G_2, F_4, E_6, E_7 , or E_8 , the split classes of W were classified (this was built on the work of Carter [Ca]), the simple ungraded $|\mathbb{C}W^-|$ -characters were constructed, and the spin character tables were computed by Morris [Mo1].

By the foundational work in the module theory of superalgebras developed by Józefiak [Joz1] (also cf. [CW, Chapter 3]), the numbers of even and odd split conjugacy classes determine the numbers of simple $\mathbb{C}W^-$ -modules of type \mathbb{M} and type \mathbb{Q} . So, we need to determine which split conjugacy classes given by Read and Morris are even or odd. Fortunately, the parity of a split class can be determined easily by the parity of the number of generators in a representative of the given split class.

In this paper we classify the simple $\mathbb{C}W^-$ -modules, not just the ungraded simple ones. This turns out to be a subtle problem which requires a combination of ideas and approaches case-by-case. To that end, we establish some structure theorems for the superalgebras $\mathbb{C}W^-$ in type B_n and type D_n . More precisely, we establish the superalgebra isomorphisms (see Theorems 4.1 and 6.2):

$$(1.4) \quad \mathbb{C}B_n^- \xrightarrow{\cong} \mathcal{C}l_n \otimes \mathbb{C}S_n \quad (\forall n), \quad \mathbb{C}D_n^- \xrightarrow{\cong} \mathcal{C}l_n^0 \otimes \mathbb{C}S_n \quad (n \text{ odd}),$$

where we denote $\mathcal{C}l_n^0$ the even subalgebra of $\mathcal{C}l_n$ (we also formulate a conjecture on $\mathbb{C}D_n^-$ for n even). The first isomorphism in (1.4) is obtained by reinterpreting [KW3, Theorem 1] for S_n , where the role of $\mathbb{C}B_n^-$ was not suspected. The construction and classification of simple $\mathbb{C}W^-$ -modules immediately follow from such superalgebra isomorphisms. For D_n with n even, we find a simple argument to upgrade Read's results [Re2].

We in addition calculate the characters of all simple $\mathbb{C}B_n^-$ -modules, and establish a characteristic map similar to the one by Frobenius which relates symmetric group representations to the ring of symmetric functions. We also provide similar construction and classification for the superalgebra $\mathfrak{H}_{B_n}^c$. This allows us to compute a simple precise formula for the spin fake degrees in type B_n in terms of a specialization of the super Schur functions and also in terms of hook lengths and contents of a Young diagram; see Theorem 4.10 and Theorem 5.8.

Note that $\mathbb{C}D_n^-$ can be regarded naturally as a subalgebra of $\mathbb{C}B_n^-$; see [KW3, 4.1]. We determine in a precise way how each simple $\mathbb{C}B_n^-$ -module decomposes upon restriction into the simple $\mathbb{C}D_n^-$ -modules, depending on whether n is odd or even. With this available, the spin fake degrees of type D_n can then be derived from those of type B_n ; see Theorems 6.13 and 7.5.

For the exceptional Weyl groups, we classify the simple $\mathbb{C}W^-$ -modules and determine their types, refining the classification results of ungraded irreducibles due to Morris [Mo1]. Then we write new code using CHEVIE [CHE] and [GAP] to compute the spin fake degrees in all exceptional cases, where we use the spin character tables computed in [Mo1] (in which we note a typo in the E_8 spin character table). The spin fake degrees of all exceptional types are tabulated in Section 9.

All the spin fake degrees for all Weyl groups are shown to be palindromic. A similar palindromicity was observed for the usual fake degrees by Beynon-Lusztig [BL].

1.5. Connections. The formulation of spin coinvariant algebras (1.3) associated to the spin Weyl groups \widetilde{W} has its origin in [KW1], where the main goal was to develop spin Hecke algebras [Wa, KW1, KW2, KW3]. The spin (affine nil) Hecke algebras have recently played a basic role in categorification of quantum supergroups. The same double covers \widetilde{W} also featured naturally in the recent work of Ciubotaru together with Barbasch and Trapa (see [BCT, Ciu]) in connection with Springer correspondence and affine Hecke algebras. It would be very interesting to understand why exactly the same spin Weyl groups appear in so diverse settings and to develop any possible connections.

In Lusztig's work [Lu2], the fake degrees were related to the generic degrees arising from Hecke algebras. In type A , the generic degrees coincide with the fake degrees [Stn]. Recently, the spin generic degrees were formulated and computed in terms of (quantum) spin Hecke algebras of type A [WW3], and they were shown to coincide with the spin fake degrees of S_n (computed in [WW1]). The quantum spin Hecke algebras beyond type A have yet to be formulated.

There has been various works (see Broué-Malle-Michel [BMM] and references therein) attempting to generalize to the setting of complex reflection groups various connections among Weyl groups, Hecke algebras and finite groups of Lie type. Our work can be formally regarded as a step toward generalization in the direction of spin Weyl groups.

1.6. Organization. The paper is organized as follows.

The preliminary Section 2 reviews the double covers \widetilde{W} of Weyl groups and some basics on the module theory of superalgebras.

In Section 3, we formulate the spin coinvariant algebras, and define the spin fake degrees for the superalgebras $\mathbb{C}W^-$ and \mathfrak{H}_W^ϵ . We formulate Morita super-equivalence of superalgebras, and show that the basic spin $\mathbb{C}W^-$ -module corresponds to the \mathfrak{H}_W^ϵ -module $\mathcal{C}l_V$ via a Morita super-equivalence.

In Section 4, the first isomorphism for $\mathbb{C}B_n^-$ in (1.4) is established. We construct and classify the simple $\mathbb{C}B_n^-$ -modules, compute their characters, and establish a characteristic map. Then we reduce the computation of the spin fake degrees for simple $\mathbb{C}B_n^-$ -modules to their counterparts for simple \mathfrak{H}_W^ϵ -modules, which is carried out in Section 5.

In Section 6 where n is set to be odd, the second isomorphism for $\mathbb{C}D_n^-$ in (1.4) is established. The simple $\mathbb{C}D_n^-$ -modules are constructed and classified. The relation between simple modules of $\mathbb{C}B_n^-$ and $\mathbb{C}D_n^-$ is worked out precisely. This allows us to reduce the computation of spin fake degrees for D_n to the counterparts for B_n .

Section 7 on spin fake degrees of D_n for n even is the counterpart of Section 6 (which was for n odd), though the detail depends much on the parity of n .

The simple $\mathbb{C}W^-$ -modules are classified and the spin fake degrees of the exceptional Weyl groups are computed in Section 8 using the spin Molien's formula given in Section 3; the spin fake degrees of exceptional types are tabulated in Section 9.

Convention. An (associative) *superalgebra* is a \mathbb{Z}_2 -graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that $A_i A_j \subseteq A_{i+j}$. A *module* over a superalgebra A is always understood in this paper as a \mathbb{Z}_2 -graded A -module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ whose grading is compatible with the action of A , i.e. $A_i M_j \subseteq M_{i+j}$. We shall denote by $|A|$ the underlying algebra of A with \mathbb{Z}_2 -grading forgotten, and by $|M|$ the $|A|$ -module.

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2. THE PRELIMINARIES

In this preliminary section, we review various known facts and set up notations for later use.

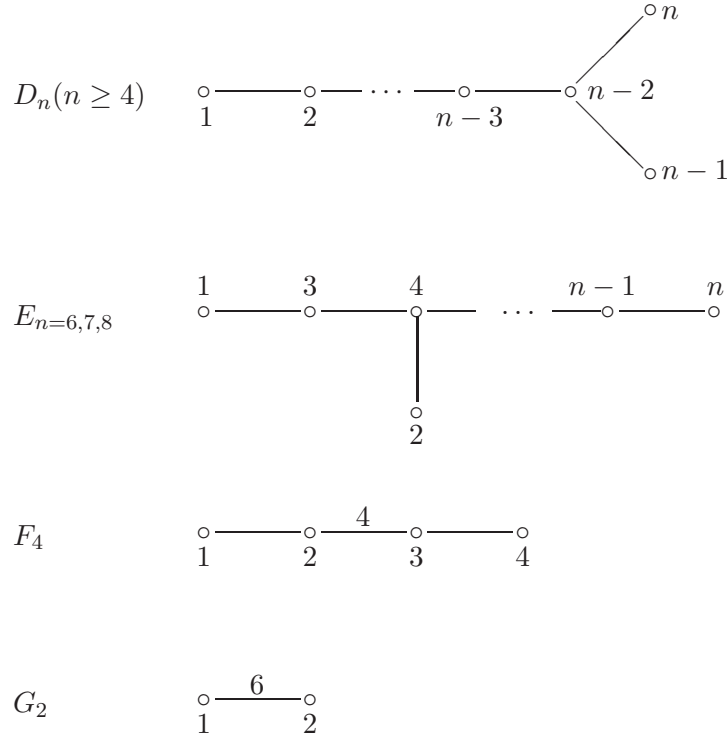
2.1. Weyl groups. Let W be an (irreducible) finite Weyl group with the following presentation:

$$(2.1) \quad \langle s_1, \dots, s_n | (s_i s_j)^{m_{ij}} = 1, m_{ii} = 1, m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{ for } i \neq j \rangle.$$

For a Weyl group W , the integers m_{ij} take values in $\{1, 2, 3, 4, 6\}$, and they are specified by the following Coxeter-Dynkin diagrams whose vertices correspond to the generators of W . By convention, we only mark the edge connecting i, j with $m_{ij} \geq 4$. We have $m_{ij} = 3$ for $i \neq j$ connected by an unmarked edge, and $m_{ij} = 2$ if i, j are not connected by an edge.

$$A_n \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ \\ 1 & & 2 & & & & n-1 & & n \end{array}$$

$$B_n(n \geq 2) \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \overset{4}{\circ} & \text{---} & \circ \\ 1 & & 2 & & & & n-1 & & n \end{array}$$



2.2. Spin Weyl groups. The Schur multipliers for finite Weyl groups W have been computed by Ihara and Yokonuma [IY]. The explicit generators and relations for the corresponding covering groups of W can be found in Karpilovsky [Kar, Table 7.1].

In this paper (as in [KW1]), we shall be concerned exclusively with a distinguished double covering \widetilde{W} of W :

$$(2.2) \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W} \xrightarrow{\theta} W \longrightarrow 1.$$

We denote by $\mathbb{Z}_2 = \{1, z\}$, and by \tilde{t}_i a fixed preimage of the generators s_i of W for each i . The group \widetilde{W} is generated by $z, \tilde{t}_1, \dots, \tilde{t}_n$ with relations

$$(2.3) \quad z^2 = 1, \quad (\tilde{t}_i \tilde{t}_j)^{m_{ij}} = \begin{cases} 1, & \text{if } m_{ij} = 1, 3 \\ z, & \text{if } m_{ij} = 2, 4, 6. \end{cases}$$

The quotient algebra of $\mathbb{C}\widetilde{W}$ by the ideal generated by $z + 1$ is denoted by $\mathbb{C}W^-$ and called the *spin Weyl group algebra* associated to W . Denote by $t_i \in \mathbb{C}W^-$ the image of \tilde{t}_i . The spin Weyl group algebra $\mathbb{C}W^-$ has the following uniform presentation: $\mathbb{C}W^-$ is the algebra generated by $t_i, 1 \leq i \leq n$, subject to the relations

$$(2.4) \quad (t_i t_j)^{m_{ij}} = (-1)^{m_{ij}+1}.$$

The algebra $\mathbb{C}W^-$ has a natural superalgebra structure by letting each t_i be odd.

Example 2.1. Let W be the Weyl group of type A_n, B_n , or D_n . Then the spin Weyl group algebra $\mathbb{C}W^-$ is generated by t_1, \dots, t_n with the labeling as in the Coxeter-Dynkin diagrams and the explicit relations summarized in the following Table A.

Table A: Relations for classical spin Weyl group algebras

Type of W	Defining Relations for $\mathbb{C}W^-$
A_n	$t_i^2 = 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \text{ if } 1 \leq i \leq n,$ $(t_i t_j)^2 = -1 \text{ if } i - j > 1$
B_n	t_1, \dots, t_{n-1} satisfy the relations for $\mathbb{C}W_{A_{n-1}}^-$, $t_n^2 = 1, \quad (t_i t_n)^2 = -1 \text{ if } i \neq n-1, n,$ $(t_{n-1} t_n)^4 = -1$
D_n	t_1, \dots, t_{n-1} satisfy the relations for $\mathbb{C}W_{A_{n-1}}^-$, $t_n^2 = 1, \quad (t_i t_n)^2 = -1 \text{ if } i \neq n-2, n,$ $t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n$

2.3. Clifford algebra. Denote by V the irreducible reflection representation of dimension n of the Weyl group W (which is the Cartan subalgebra of the corresponding simple Lie algebra). In the case of type A_n , W is the symmetric group S_{n+1} .

Note that V carries a W -invariant nondegenerate bilinear form (\cdot, \cdot) , and let $\mathcal{C}l_V$ be the Clifford algebra associated to $(V, (\cdot, \cdot))$. Denote by β_i the generator of $\mathcal{C}l_V$ corresponding to the simple root α_i normalized with $\beta_i^2 = 1$. Note that $\mathcal{C}l_V$ is naturally a superalgebra with each β_i being odd. We identify V with a suitable subspace of \mathbb{C}^m (for values of m see Table B below), and then describe the simple roots $\{\alpha_i\}$ for \mathfrak{g} using a standard orthonormal basis $\{e_i\}$ of \mathbb{C}^m . It follows that $(\alpha_i, \alpha_j) = -2 \cos(\pi/m_{ij})$. Let $\mathcal{C}l_m$ denote the Clifford algebra of \mathbb{C}^m which is generated by c_1, \dots, c_m subject to the relations

$$(2.5) \quad c_i^2 = 1, \quad c_i c_j = -c_j c_i \text{ if } i \neq j.$$

(Here c_i corresponds to the basis element e_i .) It is convenient to identify $\mathcal{C}l_V$ as a subalgebra of $\mathcal{C}l_m$ (see Table B); we may also identify $\mathcal{C}l_V$ with $\mathcal{C}l_n$ and shall do so whenever convenient.

Table B: Generators for Clifford algebra \mathcal{Cl}_V

Type of W	m	Generators for \mathcal{Cl}_V
A_n	$n+1$	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n$
B_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n-1, \beta_n = c_n$
D_n	n	$\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n-1, \beta_n = \frac{1}{\sqrt{2}}(c_{n-1} + c_n)$
E_8	8	$\beta_1 = \frac{1}{2\sqrt{2}}(c_1 + c_8 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7)$ $\beta_2 = \frac{1}{\sqrt{2}}(c_1 + c_2), \beta_i = \frac{1}{\sqrt{2}}(c_{i-1} + c_{i-2}), 3 \leq i \leq 8$
E_7	8	the subset of β_i in $E_8, 1 \leq i \leq 7$
E_6	8	the subset of β_i in $E_8, 1 \leq i \leq 6$
F_4	4	$\beta_1 = \frac{1}{\sqrt{2}}(c_1 - c_2), \beta_2 = \frac{1}{\sqrt{2}}(c_2 - c_3)$ $\beta_3 = c_3, \beta_4 = \frac{1}{2}(c_4 - c_1 - c_2 - c_3)$
G_2	3	$\beta_1 = \frac{1}{\sqrt{2}}(c_1 - c_2), \beta_2 = \frac{1}{\sqrt{6}}(-2c_1 + c_2 + c_3)$

The action of W on V preserves the bilinear form (\cdot, \cdot) and thus W acts as automorphisms of the algebra \mathcal{Cl}_V . This gives rise to a semi-direct product

$$\mathfrak{H}_W^\mathfrak{c} := \mathcal{Cl}_V \rtimes W,$$

which is called the *Hecke-Clifford algebra* for W . The algebra $\mathfrak{H}_W^\mathfrak{c}$ naturally inherits the superalgebra structure by letting elements in W be even and each β_i be odd.

2.4. Simple modules of superalgebras. In this subsection, we shall recall some standard facts about semisimple superalgebras from [Joz1] (cf. [Kle], or a forthcoming book [CW]).

The space of all $(r+s) \times (r+s)$ matrices, denoted by $M(r|s)$, is a superalgebra with the following grading, with the matrices expressed in (r, s) -block form:

$$M(r|s)_{\bar{0}} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\}, \quad M(r|s)_{\bar{1}} = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\}.$$

The set of $2n \times 2n$ matrices $Q(n) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right) \right\}$, with A, B arbitrary $n \times n$ matrices, is a subalgebra of the superalgebra $M(n|n)$.

Both $M(r|s)$ and $Q(n)$ are simple superalgebras. A classical theorem due to Wall states that all finite-dimensional simple associative superalgebras over \mathbb{C} are isomorphic to either $M(r|s)$ or $Q(n)$, for suitable r, s or n .

Theorem 2.2 (Super Wedderburn's Theorem). *Let A be a finite-dimensional semisimple associative superalgebra. Then*

$$A \cong \bigoplus_{i=1}^m M(r_i|s_i) \oplus \bigoplus_{j=1}^q Q(n_j).$$

As in the ungraded case, each simple A -module will be annihilated by all but one of the simple summands in Theorem 2.2. Since there are two types of simple superalgebras, there will also be two types of simple A -modules. We say that a simple A -module is of

type M if the summand which does not annihilate it is of the form $M(r_i|s_i)$, and of type Q if this summand is of the form $Q(n_j)$. The following generalization of Schur's lemma distinguishes between them.

Lemma 2.3 (Super Schur's Lemma). *Let M and L be simple A -modules. Then*

$$\dim \operatorname{Hom}_A(M, L) = \begin{cases} 1 & \text{if } M \cong L \text{ of type } M, \\ 2 & \text{if } M \cong L \text{ of type } Q, \\ 0 & \text{if } M \not\cong L. \end{cases}$$

Remark 2.4. Let A be a finite-dimensional \mathbb{C} -superalgebra. A type M simple A -module remains simple as an $|A|$ -module, while a type Q simple A -module becomes a sum of two non-isomorphic simple $|A|$ -modules; see [Joz1].

2.5. Split conjugacy classes. We now consider the conjugacy classes of W and \widetilde{W} . All the elements of a given conjugacy class have the same parity, so we can describe each conjugacy class in W as either even or odd.

Let K be a conjugacy class of W . Then $\theta^{-1}(K)$ is either a single conjugacy class of \widetilde{W} , or splits into two as $\theta^{-1}(K) = \widetilde{K} \sqcup z\widetilde{K}$; in the latter case, we say that K , \widetilde{K} , and $z\widetilde{K}$ are *split* classes. We say $x \in W$ is *split* (which actually depends on \widetilde{W}) if it belongs to a split conjugacy class. If we denote $\theta^{-1}(z) = \{\tilde{x}, z\tilde{x}\}$, x is split if and only if \tilde{x} and $z\tilde{x}$ are not conjugate in \widetilde{W} .

Proposition 2.5. [Joz1, Proposition 4.14] *The number of even split conjugacy classes of W is equal to the total number of simple $\mathbb{C}W^-$ -modules. The number of odd split conjugacy classes is equal to the number of simple $\mathbb{C}W^-$ -modules of type Q .*

3. SPIN COINVARIANT ALGEBRAS AND SPIN FAKE DEGREES

In this section we formulate the notion of spin coinvariant algebras and then the spin fake degrees. The Morita super-equivalence between the spin Weyl group algebras and the Hecke-Clifford algebras plays an essential role. We also formulate a spin version of Molien's formula for later use in computing the spin fake degrees for the exceptional Weyl groups.

3.1. Spin coinvariant algebras. Let A be a superalgebra. We shall denote by $A\text{-}\mathbf{mod}$ the category of (finite-dimensional) modules of the superalgebra A (with morphisms of degree one allowed). There is a parity reversing functor

$$\Pi : A\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod},$$

which sends $M = M_{\bar{0}} + M_{\bar{1}}$ to ΠM with $(\Pi M)_{\bar{0}} = M_{\bar{1}}$ and $(\Pi M)_{\bar{1}} = M_{\bar{0}}$. The underlying even subcategory $A\text{-}\mathbf{mod}_{\bar{0}}$, which consists of the same objects as $A\text{-}\mathbf{mod}$ but only even morphisms, is an abelian category. We define the Grothendieck group $R(A)$ of the category $A\text{-}\mathbf{mod}$ to be the \mathbb{Z} -module generated by all objects in $A\text{-}\mathbf{mod}$ subject to the following two relations: (i) $[\Pi M] = [M]$, (ii) $[M] = [L] + [N]$, for all L, M, N in $A\text{-}\mathbf{mod}$ satisfying a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with *even* morphisms.

Given two superalgebras A and B , we view the tensor product of superalgebras $A \otimes B$ as a superalgebra with multiplication defined by

$$(3.1) \quad (a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb') \quad (a, a' \in A, b, b' \in B)$$

where $|b|$ denotes the \mathbb{Z}_2 -degree of b , etc. Also, we shall use short-hand notation ab for $(a \otimes b) \in A \otimes B$, $a = a \otimes 1$, and $b = 1 \otimes b$. We recall the following isomorphism, which extends an earlier result of Sergeev [Se] and Yamaguchi [Ya] in type A (cf. e.g. [WW2]).

Proposition 3.1. [KW1, Theorem 2.4] *We have an isomorphism of superalgebras:*

$$\Phi : Cl_V \rtimes W \xrightarrow{\sim} Cl_V \otimes \mathbb{C}W^-,$$

which extends the identity map on Cl_V and sends $s_i \mapsto -\sqrt{-1}\beta_i t_i, \forall i$. The inverse map Ψ is the extension of the identity map on Cl_V which sends $t_i \mapsto \sqrt{-1}\beta_i s_i, \forall i$.

As in Section 2.3, we shall refer to $\mathfrak{H}_W^c := Cl_V \rtimes \mathbb{C}W$ as the *Hecke-Clifford superalgebra*.

Also as in Section 2.3, the Weyl group W acts on V as its reflection representation, and then on the symmetric algebra S^*V . The coinvariant algebra $(S^*V)_W = S^*V / \langle (S^*V)_+^W \rangle$, where $\langle (S^*V)_+^W \rangle$ denotes the ideal generated by the homogeneous W -invariants of positive degrees. A classical theorem of Chevalley states that $(S^*V)_W = \bigoplus_k (S^k V)_W$ is a graded regular representation of W (cf. [Hu]).

Definition 3.2. The *spin coinvariant algebra* for W is defined to be $Cl_V \otimes (S^*V)_W$.

Note that

$$Cl_V \otimes (S^*V)_W = \bigoplus_k Cl_V \otimes (S^k V)_W$$

is a graded regular representation of the Hecke-Clifford superalgebra \mathfrak{H}_W^c , where Cl_V acts by left multiplication on the first tensor factor and W acts diagonally, and this justifies the terminology in Definition 3.2. More generally, given a W -module M , $Cl_V \otimes M$ is naturally an \mathfrak{H}_W^c -module.

3.2. Morita super-equivalence. Recall that Cl_n is a simple superalgebra, with a unique (up to isomorphism) irreducible module U . The module U is of type \mathbb{M} for n even and of type \mathbb{Q} for n odd. We have $\dim U = 2^k$ for $n = 2k$ or $n = 2k - 1$.

Assume that a superalgebra isomorphism $Cl_n \otimes A \cong B$ exists for two superalgebras A and B . Then the two exact functors

$$(3.2) \quad \begin{aligned} \mathfrak{F} &\stackrel{\text{def}}{=} U \otimes - : A\text{-mod} \longrightarrow B\text{-mod}, \\ \mathfrak{G} &\stackrel{\text{def}}{=} \text{Hom}_{Cl_n}(U, -) : B\text{-mod} \longrightarrow A\text{-mod}, \end{aligned}$$

define a Morita super-equivalence in the following sense (cf. [Kle, Proposition 13.2.2]).

Proposition 3.3. *Assume that two superalgebras A, B satisfy a superalgebra isomorphism $Cl_n \otimes A \cong B$. Let $\mathfrak{F}, \mathfrak{G}$ be defined as in (3.2).*

- (1) *Suppose that n is even. Then the two functors \mathfrak{F} and \mathfrak{G} are equivalences of categories such that $\mathfrak{F} \circ \mathfrak{G} \cong \text{id}$, $\mathfrak{G} \circ \mathfrak{F} \cong \text{id}$.*
- (2) *Suppose that n is odd. Then $\mathfrak{F} \circ \mathfrak{G} \cong \text{id} \oplus \Pi$, $\mathfrak{G} \circ \mathfrak{F} \cong \text{id} \oplus \Pi$. Moreover, \mathfrak{F} induces a bijection between the isoclasses of irreducible A -modules of type \mathbb{M} and the isoclasses of irreducible B -modules of type \mathbb{Q} . Also \mathfrak{G} induces a bijection between the isoclasses of irreducible B -modules of type \mathbb{M} and the isoclasses of irreducible A -modules of type \mathbb{Q} .*

In particular, the Hecke-Clifford algebra $\mathfrak{H}_W^c = \mathcal{C}l_V \rtimes \mathbb{C}W$ and the spin Weyl group algebra $\mathbb{C}W^-$ are Morita super-equivalent by Proposition 3.1.

3.3. Basic spin module. In [Mo2], Morris studied the same double cover \widetilde{W} as in (2.2), and showed that there exists a surjective superalgebra homomorphism

$$(3.3) \quad \Omega : \mathbb{C}W^- \longrightarrow \mathcal{C}l_V, \quad t_i \mapsto \beta_i \ \forall i.$$

By pulling back the unique simple module U of the Clifford superalgebra $\mathcal{C}l_V$ via the homomorphism Ω , we obtain a distinguished $\mathbb{C}W^-$ -module, called the *basic spin module*, which we shall denote by \mathcal{B}_W . This is a natural generalization of the classical construction for $\mathbb{C}S_n^-$ due to Schur [Sch] (see [Joz3]).

The character of the basic spin module U of $\mathcal{C}l_n$ (with standard generators c_1, \dots, c_n) will be useful in later computations; we recall it here. Let $c_I = c_{i_1} \dots c_{i_p}$ be associated with an (ordered) subset $I = \{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$, and $c_\emptyset = 1$.

Proposition 3.4. [Joz2, Section 3C] *The character value of U at c_I is equal to*

$$\begin{cases} 0 & \text{if } I \neq \emptyset, \\ 2^k & \text{if } I = \emptyset, \text{ and } n = 2k \text{ or } 2k + 1. \end{cases}$$

The following property of basic spin modules plays a fundamental role in the formulation of the notion of spin fake degrees.

Theorem 3.5. *Let W be an arbitrary Weyl group, with V its irreducible reflection representation. Then*

- (1) *The basic spin $\mathbb{C}W^-$ -module \mathcal{B}_W is simple, of type M if $\dim V$ is even and of type Q if $\dim V$ is odd.*
- (2) *$\mathfrak{G}(\mathcal{C}l_V) \cong \mathcal{B}_W$ as $\mathbb{C}W^-$ -modules.*
- (3) *$\mathcal{C}l_V$ is a simple \mathfrak{H}_W^c -module always of type M .*

Proof. Since $\Omega : \mathbb{C}W^- \rightarrow \mathcal{C}l_V$ in (3.3) is surjective, the $\mathbb{C}W^-$ -module \mathcal{B}_W , as the pullback of the simple $\mathcal{C}l_V$ -module U via Ω , must be simple and its type comes from the type of the simple $\mathcal{C}l_V$ -module U , whence (1).

Part (3) follows immediately by (2) and Proposition 3.3.

So it remains to prove (2). The proof is case-by-case, and there are 2 main approaches: one via character computation and the other by dimension counting.

The first approach is to verify by a character computation that as \mathfrak{H}_W^c -modules,

$$(3.4) \quad \mathfrak{F}(\mathcal{B}_W) \cong \begin{cases} \mathcal{C}l_V, & \text{if } \dim V \text{ is even,} \\ \mathcal{C}l_V^{\oplus 2}, & \text{if } \dim V \text{ is odd.} \end{cases}$$

Indeed for type B_n , this isomorphism is a special case of Lemma 5.3 below (where λ is a one-row partition (n)). The verification for types A and D can also be read off from the proof of Lemma 5.3, since $\mathbb{C}S_n^-$ and $\mathbb{C}D_n^-$ are naturally subalgebras of $\mathbb{C}B_n^-$.

Since $\mathcal{C}l_V$ is a simple superalgebra with simple module U , $\mathfrak{G}(\mathcal{C}l_V) = \text{Hom}_{\mathcal{C}l_V}(U, \mathcal{C}l_V)$ has dimension equal to $\dim U$ (which is the same as $\dim \mathcal{B}_W$). Then Part (2) for a given

Weyl group W is valid if the following holds for W :

$$(3.5) \quad \mathcal{B}_W \text{ is the unique simple } \mathbb{C}W^- \text{-module of minimal dimension,} \\ \text{and the minimal dimension is equal to } \dim U.$$

It turns out (3.5) holds for exceptional Weyl groups E_6, E_7, E_8 , according to the degrees of all spin simple characters computed by Morris [Mo1]; alternatively, it can be read off from Table 3 to Table 5 in Section 9. Actually it can also be easily observed that (3.5) holds for B_n and D_n from the construction and classification of the simple $\mathbb{C}W^-$ -modules in later sections (see Propositions 4.3, 6.10, 7.3). This gives a second proof in types B_n and D_n . But we do not know how to check (3.5) directly for type A_n , though the degrees of simple characters are well known since Schur (cf. [Joz2]).

However, (3.5) is not true for G_2 and F_4 . In these two cases, we verify (3.4) by a direct character computation as follows. We shall freely use [Mo1] and our Section 8 below on exceptional Weyl groups.

The three simple spin characters of $\mathbb{C}G_2^-$ are all of degree 2 and type M, and they have different values on the conjugacy class with admissible diagram G_2 (for which we can choose t_1t_2 as a representative; cf. [Ca, Section 3, ex. (ii)]). Hence, to show (3.4) (with $\dim V = 2$) it suffices to check that both sides of (3.4) take the same (nonzero) character value at s_1s_2 . We refer to Table B for formulas of β_1 and β_2 . The action of s_1s_2 on $\mathfrak{F}(\mathcal{B}_W) = U \otimes \mathcal{B}_W$ is given by $\Phi(s_1s_2) = \beta_1\beta_2 \cdot t_1t_2$. The trace of $\beta_1\beta_2 = -\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{12}}(c_1c_2 - c_1c_3 + c_2c_3)$ on U is $-\sqrt{3}$. Since $\Omega(t_1t_2) = \beta_1\beta_2$, we see that the trace of t_1t_2 on \mathcal{B}_W is also $-\sqrt{3}$. (Note this is the opposite of the value given in [Mo1, Table VI], as we have made a different choice of $\mathbb{C}\tilde{G}_2$ conjugacy class in the preimage of the $\mathbb{C}G_2^-$ conjugacy class in question.) Hence the character value of s_1s_2 on $\mathfrak{F}(\mathcal{B}_W)$ is $(-\sqrt{3})^2 = 3$.

On the other hand, the character of s_1s_2 on $\mathcal{C}l_V$ is also 3 by the following computation:

$$\begin{aligned} s_1s_2.1 &= 1, \\ s_1s_2.\beta_1 &= s_1(\beta_1 + \sqrt{3}\beta_2) = 2\beta_1 + \sqrt{3}\beta_2, \\ s_1s_2.\beta_2 &= -s_1\beta_2 = -\sqrt{3}\beta_1 - \beta_2, \\ s_1s_2.\beta_1\beta_2 &= \beta_1\beta_2. \end{aligned}$$

Hence (3.4) holds for G_2 .

Now consider the case of F_4 . Following Morris [Mo1] and Read [Re1], there are two simple spin characters of $\mathbb{C}F_4^-$ of minimal degree 4 (both of type M); they have opposite character values on the conjugacy class with admissible diagram B_2 . Read [Re1, Table 1] provides the representative element t_2t_3 for this conjugacy class, and we will use this element to compute character values.

We identify $\mathcal{C}l_V$ with $\mathcal{C}l_4$, and refer to Table B for formulas of β_i . Since $\Omega(t_2t_3) = \beta_2\beta_3$, we see that the trace of t_2t_3 on \mathcal{B}_{F_4} is equal to the trace of $\beta_2\beta_3 = \frac{1}{\sqrt{2}}c_2c_3 - \frac{1}{\sqrt{2}}$ on U , which is $-2\sqrt{2}$. (Note this is the opposite of the value given in [Mo1, Table VII], as we have made a different choice of the split conjugacy class in question. It is, however,

the same as the value given in [Re1, Table 1].) Thus the character of $\mathfrak{F}(\mathcal{B}_{F_4}) = U \otimes \mathcal{B}_{F_4}$ on $t_2 t_3$ is $(-2\sqrt{2}) \dim U = -8\sqrt{2}$.

On the other hand, we consider

$$\Psi(t_2 t_3) = -\beta_2 s_2 \beta_3 s_3 = -(\beta_2 \beta_3 + \sqrt{2}) s_2 s_3 \in \mathfrak{H}_{F_4}^c.$$

A direct yet lengthy computation shows that the matrix of the operator $\Psi(t_2 t_3)$ acting on $\mathcal{C}l_V$ with respect to the ordered basis

$$\{1, \beta_1, \beta_2, \beta_3, \beta_4, \beta_1 \beta_2, \beta_1 \beta_3, \beta_1 \beta_4, \beta_2 \beta_3, \beta_2 \beta_4, \\ \beta_3 \beta_4, \beta_1 \beta_2 \beta_3, \beta_1 \beta_2 \beta_4, \beta_1 \beta_3 \beta_4, \beta_2 \beta_3 \beta_4, \beta_1 \beta_2 \beta_3 \beta_4\}$$

has its diagonal given by

$$\text{diag}(-\sqrt{2}, -\sqrt{2}, -\sqrt{2}, 0, -\sqrt{2}, -\sqrt{2}, 0, -\sqrt{2}, 0, -\sqrt{2}, 0, 0, -\sqrt{2}, 0, 0, 0).$$

Thus the character of $\mathcal{C}l_V$ on $\Psi(t_2 t_3)$ is $-8\sqrt{2}$, agreeing with $\mathfrak{F}(\mathcal{B}_{F_4})$. Hence (3.4) holds for F_4 .

The proof of (2) and hence of the proposition is now completed. \square

Remark 3.6. Let $W = S_n$. If we choose to work with the reflection representation \mathbb{C}^n which is not irreducible as in [Se, Ya, KW1, Wa], $\mathcal{C}l_n$ is a simple $(\mathcal{C}l_n \times S_n)$ -module, now of type Q . Theorem 3.5 in such a setting was stated without proof in [WW1].

3.4. A multiplicity identity. Let M be a W -module, E a $\mathbb{C}W^-$ -module, and F an \mathfrak{H}_W^c -module. Then the tensor product $E \otimes M$ is a $\mathbb{C}W^-$ -module under the action

$$(3.6) \quad t_i(u \otimes x) = (t_i u) \otimes (s_i x) \quad 1 \leq i \leq n, u \in E, x \in M.$$

Additionally, the tensor product $F \otimes M$ is an \mathfrak{H}_W^c -module via

$$(3.7) \quad \beta_i(u \otimes x) = (\beta_i u) \otimes x, \quad s_i(u \otimes x) = (s_i u) \otimes (s_i x)$$

for $1 \leq i \leq n, u \in F, x \in M$.

The following tensor identity is a straightforward generalization of [WW1, Lemma 3.1], and it can be proved in the same way.

Lemma 3.7. *Let M be a W -module. Then there is a $\mathbb{C}W^-$ -module isomorphism*

$$\mathfrak{G}(\mathcal{C}l_V) \otimes M \cong \mathfrak{G}(\mathcal{C}l_V \otimes M);$$

that is, $\text{Hom}_{\mathcal{C}l_V}(U, \mathcal{C}l_V) \otimes M \cong \text{Hom}_{\mathcal{C}l_V}(U, \mathcal{C}l_V \otimes M)$.

Using the Morita super-equivalence of Proposition 3.3 in the context of Proposition 3.1, we can relate the multiplicity problem for a $\mathbb{C}W^-$ -module and that for a \mathfrak{H}_W^c -module as follows. We will abuse notation to sometimes use module and character names interchangeably, and denote a module and its associated character by the same notation.

Proposition 3.8. *Suppose M is a W -module. Let χ be a simple \mathfrak{H}_W^c -character, and χ^- the corresponding simple $\mathbb{C}W^-$ -character under the Morita super-equivalence. Let $m_\chi = \dim \text{Hom}_{\mathfrak{H}_W^c}(\chi, \mathcal{C}l_V \otimes M)$ and $m_\chi^- = \dim \text{Hom}_{\mathbb{C}W^-}(\chi^-, \mathcal{B}_W \otimes M)$. Then*

$$m_\chi^- = \begin{cases} m_\chi & \text{if } n \text{ is even,} \\ 2m_\chi & \text{if } n \text{ is odd and } \chi \text{ is of type } M, \\ m_\chi & \text{if } n \text{ is odd and } \chi \text{ is of type } Q. \end{cases}$$

Proof. By definition and Lemma 2.3, we have

$$(3.8) \quad Cl_V \otimes M = \bigoplus_{\chi \text{ type } \mathfrak{m}} m_\chi \chi \oplus \bigoplus_{\chi \text{ type } \mathfrak{q}} \frac{1}{2} m_\chi \chi,$$

and

$$(3.9) \quad \mathcal{B}_W \otimes M = \bigoplus_{\chi^- \text{ type } \mathfrak{m}} m_\chi^- \chi^- \oplus \bigoplus_{\chi^- \text{ type } \mathfrak{q}} \frac{1}{2} m_\chi^- \chi^-.$$

Hence, by Proposition 3.3, Theorem 3.5, Lemma 3.7 and (3.8),

$$\begin{aligned} \mathcal{B}_W \otimes M &= \mathfrak{G}(Cl_V) \otimes M \\ &\cong \mathfrak{G}(Cl_V \otimes M) \\ &\cong \bigoplus_{\chi \text{ type } \mathfrak{m}} m_\chi \mathfrak{G}(\chi) \oplus \bigoplus_{\chi \text{ type } \mathfrak{q}} \frac{1}{2} m_\chi \mathfrak{G}(\chi) \\ &\cong \begin{cases} \bigoplus_{\chi \text{ type } \mathfrak{m}} m_\chi \chi^- \oplus \bigoplus_{\chi \text{ type } \mathfrak{q}} \frac{1}{2} m_\chi \chi^- & \text{if } n \text{ even} \\ \bigoplus_{\chi \text{ type } \mathfrak{m}} m_\chi \chi^- \oplus \bigoplus_{\chi \text{ type } \mathfrak{q}} m_\chi \chi^- & \text{if } n \text{ odd.} \end{cases} \end{aligned}$$

Comparing this with (3.9) gives the result desired. \square

3.5. Spin fake degrees. Let χ be a simple character of \mathfrak{H}_W^ϵ , and let χ^- be a simple character of $\mathbb{C}W^-$ corresponding to χ under the Morita super-equivalence as in Proposition 3.3. Define

$$(3.10) \quad \begin{aligned} P_W(\chi, t) &= \sum_k \dim \operatorname{Hom}_{\mathfrak{H}_W^\epsilon}(\chi, Cl_V \otimes (S^k V)_W) t^k, \\ P_W^-(\chi^-, t) &= \sum_k \dim \operatorname{Hom}_{\mathbb{C}W^-}(\chi^-, \mathcal{B}_W \otimes (S^k V)_W) t^k; \end{aligned}$$

$$(3.11) \quad \begin{aligned} H_W(\chi, t) &= \sum_k \dim \operatorname{Hom}_{\mathfrak{H}_W^\epsilon}(\chi, Cl_V \otimes S^k V) t^k, \\ H_W^-(\chi^-, t) &= \sum_k \dim \operatorname{Hom}_{\mathbb{C}W^-}(\chi^-, \mathcal{B}_W \otimes S^k V) t^k. \end{aligned}$$

We will refer to $H_W(\chi, t)$ informally as the graded multiplicity of χ in the \mathfrak{H}_W^ϵ -module $Cl_V \otimes S^* V$, and refer to $H_W^-(\chi^-, t)$ as the graded multiplicity of χ^- in the $\mathbb{C}W^-$ -module $\mathcal{B}_W \otimes S^* V$.

Definition 3.9. $P_W(\chi, t)$ is called the *spin fake degree* of the simple \mathfrak{H}_W^ϵ -character χ , and $P_W^-(\chi^-, t)$ is called the *spin fake degree* of the simple $\mathbb{C}W^-$ -character χ^- .

Remark 3.10. The fake degrees of a Weyl group W are the graded multiplicities of simple W -modules in the coinvariant algebra of W (cf. Lusztig [Lu2]).

The spin coinvariant algebra and spin fake degrees for the symmetric group S_n were first formulated and computed by Wan and the second author [WW1], and the terminology of spin fake degrees first appeared in [WW2].

Let d_1, \dots, d_n be the degrees of the Weyl group W (cf. [Hu, Lu2]); we recall their values in the classical cases in Table C.

Table C: Degrees of classical Weyl groups W

Type	A_n	B_n	D_n
Degrees	$2, 3, \dots, n+1$	$2, 4, \dots, 2n$	$2, 4, \dots, 2n-2, n$

Recall (cf. [Hu]) the algebra of W -invariants in S^*V is a polynomial algebra whose Hilbert polynomial is given by

$$(3.12) \quad H((S^*V)^W, t) = \frac{1}{\prod_{i=1}^n (1 - t^{d_i})}.$$

Lemma 3.11. *Let χ be a simple \mathfrak{H}_W^ϵ -character and χ^- be a simple $\mathbb{C}W^-$ -character. Then $P_W(\chi, t) = H_W(\chi, t) \prod_{i=1}^n (1 - t^{d_i})$, and $P_W^-(\chi^-, t) = H_W^-(\chi^-, t) \prod_{i=1}^n (1 - t^{d_i})$.*

Proof. Follows by definition (see (3.10), (3.11) and (3.12)) and the classical theorem of Chevalley (cf. [Hu]) that $S^*V \cong (S^*V)^W \otimes (S^*V)_W$ as graded W -modules. \square

The main goal of this paper is to compute the spin fake degrees $P_W(\chi, t)$ and $P_W^-(\chi^-, t)$ for every Weyl group W . Lemma 3.11 allows us to do the computations for the series $H_W(\chi, t)$ and $H_W^-(\chi^-, t)$ instead. Proposition 3.8 allows us to transfer back and forth any computation between $H_W(\chi, t)$ and $H_W^-(\chi^-, t)$. The computations of all these multiplicities, which are formulated for simple (graded) $\mathbb{C}W^-$ -modules, can be readily translated into the multiplicities of simple (ungraded) $|\mathbb{C}W^-|$ -modules (with some possible factors of 2 which can be determined case-by-case).

3.6. Spin Molien's formula. Recall the double cover $\theta : \widetilde{W} \rightarrow W$ from (2.2).

Lemma 3.12. *Let ψ be a simple character of $\mathbb{C}W^-$ (or, equivalently, of \widetilde{W} where z acts as -1), and let $\tilde{x} \in \widetilde{W}$. Then $\psi(\tilde{x}) = 0$, unless $\tilde{x} \in \widetilde{W}$ is even split.*

Proof. Let M be the $(\mathbb{Z}_2$ -graded) module of \widetilde{W} underlying ψ . For \tilde{x} odd, \tilde{x} switches the even and odd parts of M , and hence the trace of \tilde{x} on M is zero, i.e., $\psi(\tilde{x}) = 0$.

If \tilde{x} is even and non-split, then \tilde{x} is conjugate to $z\tilde{x}$ by definition. So we have the trace identity on M : $\text{tr}(\tilde{x}) = \text{tr}(z\tilde{x}) = -\text{tr}(\tilde{x})$, since z acts by -1 on M . Hence again $\psi(\tilde{x}) = \text{tr}(\tilde{x}) = 0$. \square

Let t be an indeterminate. We formally set

$$S_t V = \sum_{j \geq 0} (S^j V) t^j.$$

We shall denote by $\langle x \rangle$ the conjugacy class of $x \in W$, and C_x the centralizer of x in W . The following proposition is a variation on Molien's formula, which will be useful in the case of exceptional Weyl groups later on. For a finite group G , we denote by G_*^{es} the set of even split conjugacy classes of G .

Proposition 3.13 (Spin Molien's formula). *Let W be a Weyl group, and let χ^- be a simple $\mathbb{C}W^-$ -character. Then the graded multiplicity of χ^- in the $\mathbb{C}W^-$ -module $\mathcal{B}_W \otimes S^*V$ is*

$$H_W^-(\chi^-, t) = \sum_{\langle x \rangle \in W_*^{es}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{|C_x| \det(1 - tx)},$$

where $\tilde{x} \in \widetilde{W}$ is chosen such that $x = \theta(\tilde{x})$.

Proof. The character values of the \widetilde{W} -modules \mathcal{B}_W and S^*V are all real. Using the standard \widetilde{W} -character inner product (\cdot, \cdot) , we compute

$$\begin{aligned} H_W^-(\chi^-, t) &= (\chi^-, \mathcal{B}_W \otimes S_t V) \\ &= (\chi^-, \mathcal{B}_W \otimes \sum_{j \geq 0} t^j (S^j V)) \\ &= \frac{1}{|\widetilde{W}|} \sum_{\tilde{x} \in \widetilde{W}} \chi^-(\tilde{x}) \text{tr}(\tilde{x}^{-1})|_{\mathcal{B}_W} \left(\sum_{j \geq 0} \text{tr}(x^{-1})|_{S^j V} t^j \right) \\ &= \frac{1}{2|W|} \sum_{\tilde{x} \in \widetilde{W}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{\det(1 - tx)}. \end{aligned}$$

We first reorganize this sum over conjugacy classes $\langle \tilde{x} \rangle$ of \widetilde{W} . By Lemma 3.12, we may restrict $\langle \tilde{x} \rangle$ to even split classes from now on. The split conjugacy classes in \widetilde{W} are obtained as half of the inverse image of split conjugacy classes in W , that is, $\langle \tilde{x} \rangle \sqcup z\langle \tilde{x} \rangle = \theta^{-1}(\langle x \rangle)$. The classes $\langle \tilde{x} \rangle$ and $z\langle \tilde{x} \rangle$ are of the same size (as the class $\langle x \rangle$), but have opposite character values. We may reorganize the sum now over even split conjugacy classes of W . The details are as follows:

$$\begin{aligned} \frac{1}{2|W|} \sum_{\tilde{x} \in \widetilde{W}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{\det(1 - tx)} &= \frac{1}{2} \sum_{\langle \tilde{x} \rangle \in \widetilde{W}_*^{es}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{|C_x| \det(1 - tx)} \\ &= \sum_{\langle x \rangle \in W_*^{es}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{|C_x| \det(1 - tx)} \end{aligned}$$

The proposition is proved. \square

We have the following corollary of Proposition 3.13 thanks to Lemma 3.11.

Corollary 3.14. *Let W be a Weyl group with degrees d_1, \dots, d_n , and let χ^- be a simple $\mathbb{C}W^-$ -character. Then the spin fake degree of χ^- is*

$$P_W^-(\chi^-, t) = \sum_{\langle x \rangle \in W_*^{es}} \frac{\chi^-(\tilde{x}) \text{tr}(\tilde{x})|_{\mathcal{B}_W}}{|C_x| \det(1 - tx)} \cdot \prod_{i=1}^n (1 - t^{d_i}),$$

where $\tilde{x} \in \widetilde{W}$ is chosen such that $x = \theta(\tilde{x})$.

3.7. Palindromicity. Let us summarize a symmetry property shared by all spin fake degrees for all Weyl groups below. We thank Ching Hung Lam for a very helpful remark.

Theorem 3.15. *For any Weyl group W , the spin fake degrees for $\mathbb{C}W^-$ are palindromic. More precisely, for every irreducible character χ^- of $\mathbb{C}W^-$, we have*

$$P_W^-(\chi^-, t) = t^N P_W^-(\chi^-, t^{-1}),$$

where N is the number of reflections in the Weyl group W .

The values of N here for each Weyl group are recalled in Table D below.

Table D: Number N of reflections in W

Type	A_n	B_n	D_n	E_6	E_7	E_8	F_4	G_2
N	$n(n+1)/2$	n^2	$n(n-1)$	36	63	120	24	6

Proof. The proof requires a case-by-case check; see Propositions 4.11, 6.14, 7.6, and 8.4, for cases other than type A ; they rely on the explicit formulas of spin fake degrees in these cases, which we will compute in the subsequent sections.

Now let $W = S_n$ in type A_{n-1} . The simple $\mathbb{C}S_n^-$ -modules are parameterized by strict partitions λ of n , and the corresponding spin fake degrees $P_{S_n}^-(\lambda, t)$ were computed in [WW1, Theorem A] (cf. [WW2]). The formula for $P_{S_n}^-(\lambda, t)$ is in terms of $n(\lambda)$ given in (4.5), contents c_\square , and shifted hook lengths h_\square^* , and we refer to *loc. cit.* for detailed definitions. It is actually clear that $P_{S_n}^-(\lambda, t) = t^a P_{S_n}^-(\lambda, t^{-1})$ for some shift integer a , since one observes that $P_{S_n}^-(\lambda, t)$ is a product of factors of the form $(1 \pm t^*)^{\pm 1}$. So it remains to determine the shift number a , which is the sum of the highest and the lowest powers of t appearing in $P_{S_n}^-(\lambda, t)$. Thus

$$\begin{aligned} a &= 2n(\lambda) + \frac{n^2 + n - 2}{2} + \sum_{\square \in \lambda^*} (c_\square - h_\square^*) \\ &= 2n(\lambda) + \frac{n^2 + n - 2}{2} - (2n(\lambda) + n - 1) = \frac{n^2 - n}{2}, \end{aligned}$$

which is the number of reflections in S_n . \square

Remark 3.16. A similar palindromicity property holds for the usual fake degrees, see Beynon-Lusztig [BL, Proposition A], which can be regarded as a variant of Poincare duality. However, the shift numbers for the usual fake degrees depend on the irreducible characters (as well as on the Weyl groups).

4. THE SPIN FAKE DEGREES OF TYPE B_n

4.1. Structure of the algebra $\mathbb{C}B_n^-$. We shall simply write the Weyl group of type B_n as B_n , its double cover as \tilde{B}_n , and the spin Weyl group algebra as $\mathbb{C}B_n^-$. Recall the generators β_i for $\mathcal{C}l_V$ from Table B, where V is the reflection representation of a Weyl group W (in this case B_n), and note that we can identify $\mathcal{C}l_V$ and $\mathcal{C}l_n$. The following is a new formulation of Khongsap-Wang [KW3, Theorem 1] in the case of S_n , which now describes the structure of the superalgebra $\mathbb{C}B_n^-$.

Theorem 4.1. *There is an isomorphism of superalgebras*

$$\phi^B : \mathbb{C}B_n^- \xrightarrow{\cong} \mathcal{C}l_n \otimes \mathbb{C}S_n,$$

$$t_i \mapsto \begin{cases} \beta_i s_i & \text{if } i \leq n-1, \\ c_n & \text{if } i = n. \end{cases}$$

The inverse map sends $s_i \mapsto \beta_i t_i$ and $c_i \mapsto t_i t_{i+1} \cdots t_{n-1} t_n t_{n-1} \cdots t_{i+1} t_i$ for all possible i . (Note each s_i is even here.)

Proof. Following [KW3, Theorem 1] we denote by $\mathcal{C}l_n \rtimes_- \mathbb{C}S_n^-$ the superalgebra generated by $\mathcal{C}l_n$ and $\mathbb{C}S_n^-$ with the additional relation that $t_i c_j = -c_j^{s_i} t_i$ for all i, j . Our simple yet new observation here is that there is a natural identification of superalgebras

$$(4.1) \quad \mathbb{C}B_n^- \xrightarrow{\cong} \mathcal{C}l_n \rtimes_- \mathbb{C}S_n^-$$

by sending $t_i \mapsto t_i$ for $i = 1, \dots, n-1$, and $t_n \mapsto c_n$. To that end, it suffices to check that $t_{n-1} c_n t_{n-1} c_n = -c_n t_{n-1} c_n t_{n-1}$ in $\mathcal{C}l_n \rtimes_- \mathbb{C}S_n^-$. Indeed,

$$\begin{aligned} t_{n-1} c_n t_{n-1} c_n &= c_{n-1} t_{n-1} c_{n-1} t_{n-1} = c_{n-1} c_n t_{n-1} t_{n-1} \\ &= -c_n c_{n-1} t_{n-1} t_{n-1} = -c_n t_{n-1} c_n t_{n-1}. \end{aligned}$$

On the other hand, we have an explicit isomorphism $\mathcal{C}l_n \rtimes_- \mathbb{C}S_n^- \cong \mathcal{C}l_n \otimes \mathbb{C}S_n$, which extends the identity map on $\mathcal{C}l_n$ and sends $t_i \mapsto \beta_i s_i$ for $i \leq n-1$, by [KW3, Theorem 1] specialized for $W = S_n$. Now the theorem follows from this isomorphism and the identification (4.1). \square

4.2. Split classes for B_n . Let \mathcal{P} be the set of all partitions, \mathcal{OP} the set of all odd partitions, and \mathcal{EP} the set of all even partitions. For a partition λ of n , we write $|\lambda| = n$ and denote $\lambda \vdash n$.

With the identification $\mathbb{Z}_2 = \{+, -\}$, an element x in $B_n = \mathbb{Z}_2^n \rtimes S_n$ is a product of positive and negative cycles of various lengths. Collecting the lengths of positive (respectively, negative) cycles together gives us a partition ρ_+ (respectively, ρ_-), and we say x is of type (ρ_+, ρ_-) . It is well known [Mac, I, Appendix B] that the conjugacy classes of the group B_n are parameterized by the types of total size n . For example, the identity element of B_n has type $((1^n), \emptyset)$.

Lemma 4.2. (cf. [Re2])

- (1) *The split conjugacy classes of B_n are the classes of the following types:*
 - (a) $(\rho_+, \rho_-) \in (\mathcal{OP}, \mathcal{EP})$;
 - (b) $(\rho_+, \rho_-) \in (\emptyset, \mathcal{P})$ (only when n is odd).
- (2) *The split classes of type $(\rho_+, \rho_-) \in (\mathcal{OP}, \mathcal{EP})$ are even while those of type $(\rho_+, \rho_-) \in (\emptyset, \mathcal{P})$ are odd.*

Proof. (1) is exactly [Re2, Theorem 4.1], where Read uses the terminology α -regular to refer to split classes.

Since all generators t_i are odd, the parity of an element $t_{i_1} \cdots t_{i_k}$, and thus of its conjugacy class, is equal to the parity of k . Now (2) follows by counting the number of generators in a representative element of each conjugacy class as given in [Re2]. \square

4.3. Simple $\mathbb{C}B_n^-$ -modules. It follows from Proposition 2.5 and Lemma 4.2 that all simple modules of B_n for n even (respectively, n odd) are of type \mathbf{M} (respectively, type \mathbf{Q}). Denote by S^λ the Specht module associated to a partition λ . Recall the unique simple $\mathcal{C}l_V$ -module U . The pullback of the simple $(\mathcal{C}l_V \otimes \mathbb{C}S_n)$ -module $U \otimes S^\lambda$ via the isomorphism ϕ^B is a simple $\mathbb{C}B_n^-$ -module, which is denoted by B^λ .

Proposition 4.3. *$\{B^\lambda | \lambda \vdash n\}$ is a complete set of pairwise inequivalent simple $\mathbb{C}B_n^-$ -modules, all of type \mathbf{M} when n is even, and all of type \mathbf{Q} when n is odd.*

Proof. This follows directly from the isomorphism in Theorem 4.1; note that the $\mathbb{C}S_n$ is purely even and that the $\mathbb{C}B_n^-$ -module U is of type \mathbf{M} if and only if n is even. \square

Remark 4.4. The *ungraded* simple modules for $\mathbb{C}B_n^-$ have been classified and constructed in completely different approaches by Read [Re2, Theorem 5.1] (also cf. Stembridge [Stm, Theorem 9.2]). In light of Remark 2.4, Proposition 4.3 allows us to recover easily Read's classification of irreducible *ungraded* modules.

Next we shall determine the character of B^λ . We choose the canonical positive cycle in $\mathbb{C}B_n^-$ which permutes a through $a+k$ to be $t_a t_{a+1} \cdots t_{a+k-1}$, and the canonical negative cycle in $\mathbb{C}B_n^-$ which permutes those same elements to be $t_a t_{a+1} \cdots t_{a+k-1} b_{a+k}$, where

$$(4.2) \quad b_n = t_n, \quad b_i = t_i b_{i+1} t_i, \quad \text{for } 0 \leq i \leq n-1.$$

(In particular, $b_i \mapsto c_i$ under the isomorphism in Theorem 4.1.) The representative element for the conjugacy class of type (α, β) is the product of the corresponding canonical cycles. Let χ_μ^λ be the character value of the Specht module S^λ evaluated on elements of S_n of type μ . Given partitions α, β , we denote by $\alpha \cup \beta$ the partition obtained by collecting and rearranging the parts from α and β . By Lemma 4.2, the even split conjugacy classes of B_n are parametrized by the types $(\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})$, which are in bijection with the partitions of n by taking $\alpha \cup \beta$; we will use such an identification below whenever it is convenient.

Proposition 4.5. *The character value of the simple $\mathbb{C}B_n^-$ -module B^λ at an even split element of type $(\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})$ is*

$$\begin{cases} 2^{\ell(\alpha \cup \beta)/2} (-1)^{(n-\ell(\alpha))/2} \chi_{\alpha \cup \beta}^\lambda & \text{if } n \text{ is even,} \\ 2^{(\ell(\alpha \cup \beta)+1)/2} (-1)^{(n-\ell(\alpha))/2} \chi_{\alpha \cup \beta}^\lambda & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is mainly based on Theorem 4.1 and Proposition 4.3.

Let $x \in \mathbb{C}B_n^-$ be a representative element of type $(\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})$. When we compute the character value of B^λ at x , i.e., the character value of $U \otimes S^\lambda$ at $\phi^B(x) \in \mathcal{C}l_V \otimes \mathbb{C}S_n$ (for the isomorphism ϕ^B see Proposition 4.3), we may ignore all nontrivial products of c_i by Proposition 3.4.

We write x as a product of cycles $x = b_J \sigma$. For a positive $(k+1)$ -cycle with k even, the image of the canonical cycle is

$$\begin{aligned} \phi^B(t_a t_{a+1} \cdots t_{a+k-1}) &= \beta_a \cdots \beta_{a+k-1} s_a \cdots s_{a+k-1} \\ &= 2^{-\frac{k}{2}} (c_a c_{a+1} - c_a c_{a+2} - 1 + c_{a+1} c_{a+2}) \cdots \\ &\quad \cdot (c_{a+k-2} c_{a+k-1} - c_{a+k-2} c_{a+k} - 1 + c_{a+k-1} c_{a+k}) s_a \cdots s_{a+k-1} \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k}{2}} s_a \cdots s_{a+k-1} + (\text{terms with } c_i). \end{aligned}$$

For a negative $(k+1)$ -cycle with k odd, the image of the canonical cycle is

$$\begin{aligned} \phi^B(t_a t_{a+1} \cdots t_{a+k-1} r_{a+k}) &= \beta_a \cdots \beta_{a+k-1} c_{a+k} s_a \cdots s_{a+k-1} \\ &= 2^{-\frac{k}{2}} (c_a c_{a+1} - c_a c_{a+2} - 1 + c_{a+1} c_{a+2}) \cdots \\ &\quad \cdot (c_{a+k-3} c_{a+k-2} - c_{a+k-3} c_{a+k-1} - 1 + c_{a+k-2} c_{a+k-1}) \\ &\quad \cdot (c_{a+k-1} c_{a+k} - 1) s_a \cdots s_{a+k-1} \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k+1}{2}} s_a \cdots s_{a+k-1} + (\text{terms with } c_i). \end{aligned}$$

Multiplying these together, we have

$$\phi^B(x) = 2^{-\frac{n-\ell(\alpha \cup \beta)}{2}} (-1)^{\frac{n-\ell(\alpha)}{2}} \sigma + (\text{terms with } c_i).$$

Thus by Proposition 3.4, the character value of B^λ at x , i.e., the character value of $U \otimes S^\lambda$ at $\phi^B(x)$ is equal to

$$\begin{cases} 2^{\frac{n}{2}} 2^{-\frac{n-\ell(\alpha \cup \beta)}{2}} (-1)^{\frac{n-\ell(\alpha)}{2}} \chi_{\alpha \cup \beta}^\lambda & \text{if } n \text{ is even,} \\ 2^{\frac{n+1}{2}} 2^{-\frac{n-\ell(\alpha \cup \beta)}{2}} (-1)^{\frac{n-\ell(\alpha)}{2}} \chi_{\alpha \cup \beta}^\lambda & \text{if } n \text{ is odd,} \end{cases}$$

which is the same as given in the proposition. \square

Remark 4.6. Read [Re2] chooses representative elements which differ from ours in the use of $(-1)^{n-i} b_i$ in place of b_i , but this difference in sign does not affect the computation of characters. The character formula in Proposition 4.5 agrees with that computed by Read [Re2, Theorems 3.5, 5.1], and so our labeling of the simple (graded or ungraded) characters is consistent with Read (cf. Remark 4.4). Stembridge [Stm] used a form of the basic spin module \mathcal{B}_{B_n} resulting from $-\beta_i$ rather than β_i , and so his $\mathbb{C}B_n^-$ -modules differ from ours by a tensor with sgn .

4.4. The characteristic map for $\mathbb{C}B_n^-$. Let $R(\mathbb{C}B_n^-)$ be the Grothendieck group of the category of $\mathbb{C}B_n^-$ -modules. If we replace isomorphism classes of modules by their characters, it becomes a free abelian group with a basis made up of the irreducible characters. Now define

$$R^- = \bigoplus_{n=0}^{\infty} R(\mathbb{C}B_n^-),$$

when $R(\mathbb{C}B_0^-) = \mathbb{Z}$. Set $R_{\mathbb{Q}}^- := \mathbb{Q} \otimes_{\mathbb{Z}} R^-$.

We shall define a ring structure on R^- as follows. Let $\mathbb{C}B_{m,n}^-$ be the subalgebra of $\mathbb{C}B_{m+n}^-$ generated by $\mathbb{C}B_m^- \times \mathbb{C}B_n^-$. For a $\mathbb{C}B_m^-$ -module M and a $\mathbb{C}B_n^-$ -module N ,

$M \otimes N$ is naturally a $\mathbb{C}B_{m,n}^-$ -module, and we define the product

$$[M] \cdot [N] = [\mathbb{C}B_{m+n}^- \otimes_{\mathbb{C}B_{m,n}^-} (M \otimes N)],$$

and then extend by \mathbb{Z} -bilinearity. It follows from the properties of the induced characters that the multiplication on R^- is commutative and associative.

Given $\mathbb{C}B_n^-$ -modules M, N , we define a bilinear form on R^- by letting

$$(4.3) \quad \langle M, N \rangle = \dim \operatorname{Hom}_{\mathbb{C}B_n^-}(M, N).$$

Denote by Λ the ring of symmetric functions in infinitely many variables, which is the \mathbb{Z} -span of the monomial symmetric functions m_λ for $\lambda \in \mathcal{P}$, and let $\Lambda_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$. There is a standard bilinear form (\cdot, \cdot) on Λ such that the Schur functions s_λ form an orthonormal basis for Λ . The ring $\Lambda_{\mathbb{Q}}$ admits several other distinguished bases: the complete homogeneous symmetric functions $\{h_\lambda\}$, the elementary symmetric functions $\{e_\lambda\}$, and the power-sum symmetric functions $\{p_\lambda\}$. See [Mac].

Now define the (*spin*) *characteristic map* $\operatorname{ch}^- : R_{\mathbb{Q}}^- \rightarrow \Lambda_{\mathbb{Q}}$ as the linear map

$$(4.4) \quad \operatorname{ch}^-(\phi) = \sum_{\lambda \vdash n} z_\lambda^{-1} (-1)^{\frac{n-\ell(\alpha)}{2}} 2^{-\frac{\ell(\lambda)}{2}} \phi(\lambda) p_\lambda,$$

where $\phi \in R(\mathbb{C}B_n^-)$, $\phi(\lambda)$ is the character value of ϕ at an element of type (α, β) with $\alpha \cup \beta = \lambda$ and $\alpha \in \mathcal{OP}$ and $\beta \in \mathcal{EP}$, and z_λ is the order of the centralizer in S_n of an element of cycle type λ .

Theorem 4.7. *The characteristic map $\operatorname{ch}^- : R_{\mathbb{Q}}^- \rightarrow \Lambda_{\mathbb{Q}}$ is an isometric isomorphism of graded algebras, sending the character of B^λ to s_λ when $|\lambda|$ is even and to $\sqrt{2}s_\lambda$ when $|\lambda|$ is odd.*

Proof. Recall that the characters of the irreducible modules B^λ for $\lambda \in \mathcal{P}$, defined in Proposition 4.3, form a basis for R^- . This becomes an orthonormal basis if we divide the characters of type \mathbb{Q} modules B^λ by $\sqrt{2}$, thanks to the super version of Schur's Lemma, Lemma 2.3; this happens exactly when n is odd by Proposition 4.3.

By plugging the character values of B^λ computed in Proposition 4.3 to (4.4), the characteristic map ch^- sends the elements of this orthonormal basis to the corresponding Schur functions, so it is an isometry.

It remains to check that ch^- is an algebra homomorphism. Let $\phi \in R(\mathbb{C}B_m^-)$ and $\chi \in R(\mathbb{C}B_n^-)$, and consider the image of their product under the characteristic map. When splitting $\lambda = (\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})_{m+n}$ into partitions $\mu \vdash m, \nu \vdash n$, we will write α_μ, α_ν for the corresponding pieces of α .

$$\begin{aligned} \operatorname{ch}^-(\phi\chi) &= \sum_{\lambda \vdash m+n} z_\lambda^{-1} (-1)^{\frac{m+n-\ell(\alpha)}{2}} 2^{-\frac{\ell(\lambda)}{2}} (\phi\chi)(\lambda) p_\lambda \\ &\stackrel{(*)}{=} \sum_{\lambda \vdash m+n} z_\lambda^{-1} (-1)^{\frac{m+n-\ell(\alpha)}{2}} 2^{-\frac{\ell(\lambda)}{2}} \sum_{\substack{\mu \cup \nu = \lambda \\ \mu \vdash m, \nu \vdash n}} z_\lambda z_\mu^{-1} z_\nu^{-1} \phi(\mu) \chi(\nu) p_\lambda \\ &= \sum_{\lambda \vdash m+n} \sum_{\substack{\mu \cup \nu = \lambda \\ \mu \vdash m, \nu \vdash n}} (-1)^{\frac{m-\ell(\alpha_\mu)}{2}} (-1)^{\frac{n-\ell(\alpha_\nu)}{2}} 2^{-\frac{\ell(\mu)}{2}} 2^{-\frac{\ell(\nu)}{2}} z_\mu^{-1} z_\nu^{-1} \phi(\mu) \chi(\nu) p_\mu p_\nu \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\mu \vdash m} z_\mu^{-1} (-1)^{\frac{m-\ell(\alpha_\mu)}{2}} 2^{-\frac{\ell(\mu)}{2}} \phi(\mu) p_\mu \sum_{\nu \vdash n} z_\nu^{-1} (-1)^{\frac{n-\ell(\alpha_\nu)}{2}} 2^{-\frac{\ell(\nu)}{2}} \chi(\nu) p_\nu \\
 &= \text{ch}^-(\phi) \text{ch}^-(\chi).
 \end{aligned}$$

In the equation (\star) above, we have used a formula for the character value $(\phi\chi)(\lambda)$, which can be established in a completely analogous way as for Lemma 5.5 below. This proves the theorem. \square

4.5. Spin fake degrees of B_n . For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n , denote

$$(4.5) \quad n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i.$$

We denote by h_\square the *hook length* associated to a cell \square in the Young diagram of λ ; the *content* associated to a cell \square is defined to be the difference between the column number and the row number of \square .

Example 4.8. For $\lambda = (4, 3, 1)$, the hook lengths are listed in the corresponding cells of the left-hand diagram, and the contents in the corresponding cells of the right-hand diagram as follows:

6	4	3	1
4	2	1	
1			

0	1	2	3
-1	0	1	
-2			

In this case, $n(\lambda) = 5$.

Introduce a parity function

$$p(n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.9. *The graded multiplicity of B^λ in the $\mathbb{C}B_n^-$ -module $\mathcal{B} \otimes S^*V$ is*

$$H_{B_n}^-(\lambda, t) = 2^{p(n)} t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_\square + 1}}{1 - t^{2h_\square}}.$$

Proof. We will compute the graded multiplicities for the simple $\mathfrak{H}_{B_n}^\epsilon$ -modules K^λ in $Cl_n \otimes S^*V$ in Theorem 5.7. The theorem follows from Lemma 5.3, Theorem 5.7 and Proposition 3.8. \square

The following is equivalent to Theorem 4.9 by Lemma 3.11 and use of the well-known fact that the degrees of B_n are $2, 4, \dots, 2n$.

Theorem 4.10. *The spin fake degree of B^λ , for $\lambda \vdash n$, is*

$$P_{B_n}^-(\lambda, t) = 2^{p(n)} t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_\square + 1}}{1 - t^{2h_\square}} (1 - t^2)(1 - t^4) \cdots (1 - t^{2n}).$$

We have the following palindromicity of the spin fake degrees for B_n .

Proposition 4.11. *For $\lambda \vdash n$, we have $P_{B_n}^-(\lambda, t) = t^{n^2} P_{B_n}^-(\lambda, t^{-1})$.*

Proof. Observe that $P_{B_n}^-(\lambda, t) = t^a P_{B_n}^-(\lambda, t^{-1})$ for some integer a , since each of its factors is of the form $(1 \pm t^*)^{\pm 1}$. It remains to determine the shift number a , which is the sum of the highest power of t appearing in $P_{B_n}^-(\lambda, t)$ with nonzero coefficient, and the lowest. Thus

$$\begin{aligned} a &= 2n(\lambda) + n(n+1) + \sum_{\square \in \lambda} (2c_{\square} + 1) - 2 \sum_{\square \in \lambda} h_{\square} + 2n(\lambda) \\ &= 4n(\lambda) + n^2 + 2n + 2 \sum_{\square \in \lambda} (c_{\square} - h_{\square}) \\ &= 4n(\lambda) + n^2 + 2n + 2(n(\lambda') - n(\lambda) - (n(\lambda) + n(\lambda') + n)) = n^2. \end{aligned}$$

The proposition is proved. \square

5. THE SPIN FAKE DEGREES OF THE HECKE-CLIFFORD ALGEBRA $\mathfrak{H}_{B_n}^{\epsilon}$

In this section we will work with the Hecke-Clifford algebra in order to complete the computation of spin fake degrees of type B_n (see the proof of Theorem 4.9).

5.1. Split classes for Γ_n . We first realize the Hecke-Clifford algebra $\mathfrak{H}_{B_n}^{\epsilon}$ as a spin group algebra for a finite group Γ_n . Define the semidirect product

$$\Gamma_n := \mathbb{Z}_2^n \rtimes B_n,$$

which as a group is isomorphic to a wreath product $(\mathbb{Z}_2 \times \mathbb{Z}_2)^n \rtimes S_n$. So it is well known (see [Mac, I, Appendix B]) that its conjugacy classes \mathcal{C}_{ρ} are parametrized by quadruples of partitions $\rho = (\rho_{++}, \rho_{+-}, \rho_{-+}, \rho_{--})$ of total size n , where the second signs in the subscripts are understood to correspond to the second factor in $\mathbb{Z}_2 \times \mathbb{Z}_2$ (which has its origin from B_n). Denote by $\ell(\rho) = \ell(\rho_{++}) + \ell(\rho_{+-}) + \ell(\rho_{-+}) + \ell(\rho_{--})$.

Consider a finite group

$$\Pi_n = \langle a_1, \dots, a_n, z | a_i^2 = z^2 = 1, a_i a_j = z a_j a_i, a_i z = z a_i \rangle,$$

and write the generators of $B_n = \mathbb{Z}_2^n \rtimes S_n$ as $\tau_1, \dots, \tau_n, s_1, \dots, s_{n-1}$, where τ_i is a generator of the i th copy of \mathbb{Z}_2 . Then the semidirect product $\tilde{\Gamma}_n = \Pi_n \rtimes B_n$ is a group such that z is central, $a_j s_i = s_i a_{s_i(j)}$, and

$$\tau_i a_j = \begin{cases} z a_j \tau_i & \text{if } i = j \\ a_j \tau_i & \text{if } i \neq j. \end{cases}$$

The group $\tilde{\Gamma}_n$ is a double cover of Γ_n :

$$1 \longrightarrow \{1, z\} \longrightarrow \tilde{\Gamma}_n \xrightarrow{\theta} \Gamma_n \longrightarrow 1.$$

Introduce the spin group algebra $\mathbb{C}\Gamma_n^- := \mathbb{C}\tilde{\Gamma}_n / \langle z + 1 \rangle$. The quotient algebra $\mathbb{C}\Pi_n / \langle z + 1 \rangle$ is identified with $\mathcal{C}l_n$ by $\bar{a}_i \mapsto c_i$, which leads to a natural identification of the superalgebras

$$(5.1) \quad \mathbb{C}\Gamma_n^- \equiv \mathfrak{H}_{B_n}^{\epsilon},$$

where the superalgebra structure on $\mathbb{C}\Gamma_n^-$ is given by letting each \bar{a}_i be odd and each s_i and τ_i be even. We feel free to use the identification (5.1) below: while $\mathfrak{H}_{B_n}^{\epsilon}$ appears to be super-equivalent to $\mathbb{C}B_n^-$, $\mathbb{C}\Gamma_n^-$ allows one to appeal to finite group techniques.

The support of a (signed) permutation $\sigma \in B_n$ is $\text{supp}(\sigma) = \{i \mid 1 \leq i \leq n, \sigma(i) \neq i\}$, and for an (ordered) subset $I = \{i_1, \dots, i_r\}$, we denote $a_I = a_{i_1} \dots a_{i_r}$, and similarly for τ_I . An arbitrary element $z^* a_I \tau_J \sigma \in \tilde{\Gamma}_n$ may be written as a product

$$z^* a_I \tau_J \sigma = z^* (a_{I_1} \tau_{J_1} \sigma_1) \cdots (a_{I_k} \tau_{J_k} \sigma_k),$$

where $*$ $\in \{0, 1\}$, $\sigma = \sigma_1 \cdots \sigma_k \in S_n$ is a product of disjoint cycles, and $I_a, J_a \subseteq \text{supp}(\sigma_a)$ for all $1 \leq a \leq k$. Note that $|I| = \sum_{i=1}^k |I_i|$.

Let \mathcal{C}_ρ be a split conjugacy class of Γ_n . Then its inverse image in $\tilde{\Gamma}_n$ is $\theta^{-1}(\mathcal{C}_\rho) = \mathcal{C}_\rho^+ \sqcup z\mathcal{C}_\rho^+$. In particular, we can make sense of split classes in Γ_n and $\tilde{\Gamma}_n$ as before.

Proposition 5.1. *Let \mathcal{C}_ρ be a conjugacy class of Γ_n . Then \mathcal{C}_ρ is even split if and only if $\rho \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$.*

Proof. (\Rightarrow) Let \mathcal{C}_ρ be an even conjugacy class of Γ_n .

Case 1: Suppose $\rho_{++} \notin \mathcal{OP}$. Then ρ_{++} has at least one even part, so $\theta^{-1}(\mathcal{C}_\rho)$ contains a product of disjoint cycles of the form $a_I \tau_J \sigma = (1, \dots, r)(a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l)$, where r is even; note $|I|$ is also even, since \mathcal{C}_ρ is even. We compute the following conjugation of $a_I \tau_J \sigma$:

$$\begin{aligned} a_{1\dots r}^{-1} (a_I \tau_J \sigma) a_{1\dots r} &= a_{r\dots 1} (1, \dots, r) a_{1\dots r} z^{|I|r} (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) \\ &= a_{r\dots 1} a_{2\dots r} a_1 (1, \dots, r) (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) \\ &= z a_I \tau_J \sigma. \end{aligned}$$

Thus \mathcal{C}_ρ does not split if $\rho_{++} \notin \mathcal{OP}$.

Case 2: Suppose $\rho_{+-} \notin \mathcal{EP}$. Then ρ_{+-} has an odd part, so $\theta^{-1}(\mathcal{C}_\rho)$ contains an element of the form $a_I \tau_J \sigma = (\tau_1(1, \dots, r))(a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l)$, where r is odd; note $|I| = \sum_i |I_i|$ is even, since \mathcal{C}_ρ is even. We compute the following conjugation:

$$a_{1\dots r}^{-1} (a_I \tau_J \sigma) a_{1\dots r} = a_{r\dots 1} \tau_1(1, \dots, r) a_{1\dots r} z^{|I|r} (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) = z a_I \tau_J \sigma,$$

where we used $a_{r\dots 1} \tau_1(1, \dots, r) a_{1\dots r} = z \tau_1 a_{r\dots 1} a_{2\dots r} a_1(1, \dots, r) = z \tau_1(1, \dots, r)$.

Thus \mathcal{C}_ρ does not split if $\rho_{+-} \notin \mathcal{EP}$.

Case 3: Suppose $\rho_{-+} \neq \emptyset$. Then $\theta^{-1}(\mathcal{C}_\rho)$ contains an element $a_I \tau_J \sigma$ of the form $(a_1(1, \dots, r))(a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l)$. Note $|I|$ is even, since \mathcal{C}_ρ is even. We compute the following conjugation:

$$\begin{aligned} (a_1(1, \dots, r))^{-1} (a_I \tau_J \sigma) (a_1(1, \dots, r)) &= (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) (a_1(1, \dots, r)) \\ &= z^{|I|-1} a_1(1, \dots, r) (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) \\ &= z a_I \tau_J \sigma. \end{aligned}$$

Thus \mathcal{C}_ρ does not split if $\rho_{-+} \neq \emptyset$.

Case 4: Suppose $\rho_{--} \neq \emptyset$. Then $\theta^{-1}(\mathcal{C}_\rho)$ contains an element of the form $a_I \tau_J \sigma = (a_1 \tau_j(1 \cdots r))(a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l)$, where $j \in \{1, \dots, r\}$. Note $|I|$ is even, since \mathcal{C}_ρ is

even. We compute the following conjugation:

$$\begin{aligned}
& (a_1 \tau_j(1, \dots, r))^{-1} (a_I \tau_J \sigma) (a_1 \tau_j(1, \dots, r)) \\
&= (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) (a_1 \tau_j(1, \dots, r)) \\
&= z^{|I|-1} a_1 \tau_j(1, \dots, r) (a_{I_2} \tau_{J_2} \sigma_2) \cdots (a_{I_l} \tau_{J_l} \sigma_l) \\
&= z a_I \tau_J \sigma.
\end{aligned}$$

Thus \mathcal{C}_ρ does not split if $\rho_{--} \neq \emptyset$.

Hence, we have shown that if \mathcal{C}_ρ is even split then $\rho \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$.

(\Leftarrow) Suppose $\rho \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$. Then the conjugacy class \mathcal{C}_ρ is clearly even.

Suppose \mathcal{C}_ρ does not split, i.e. any $x \in \mathcal{C}_\rho$ is conjugate to zx . Take an element in $\theta^{-1}(\mathcal{C}_\rho)$ of the form $\tau_K \sigma = \sigma_1 \dots \sigma_p (\tau_{k_1} \sigma'_1) \cdots (\tau_{k_q} \sigma'_q)$, with each σ_i an odd cycle, each σ'_i an even cycle, $p = \ell(\rho_{++})$, $q = \ell(\rho_{+-})$, $k_i \in \text{supp}(\sigma'_i)$, and $\sigma = \sigma_1 \dots \sigma_p \sigma'_1 \dots \sigma'_q$ a product of disjoint cycles.

Since \mathcal{C}_ρ does not split, there exists $a_J t \in \Pi_n \rtimes B_n$ such that $a_J t \tau_K \sigma = z \tau_K \sigma a_J t$. Then we must have $z a_J = a_{\tau_K \sigma(J)}$, $\text{supp}(\tau_K \sigma) \subseteq J$, and $t \tau_K \sigma = \tau_K \sigma t$. On the other hand, we compute

$$\begin{aligned}
a_{\tau_K \sigma(J)} &= \tau_K \sigma a_J (\tau_K \sigma)^{-1} \\
&= \sigma_1 \dots \sigma_p (\tau_{k_1} \sigma'_1) \cdots (\tau_{k_q} \sigma'_q) a_J (\tau_K \sigma)^{-1} \\
&= z^{q+|\rho_{++}|+|\rho_{+-}|-\ell(\rho_{++})-\ell(\rho_{+-})} a_J \\
&= z^{|\rho_{++}|-\ell(\rho_{++})} a_J = a_J
\end{aligned}$$

which is a contradiction to $z a_J = a_{\tau_K \sigma(J)}$. So \mathcal{C}_ρ must split. \square

For $\rho \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$, the split class $\theta^{-1}(\mathcal{C}_\rho)$ is a disjoint union of two conjugacy classes; if we denote the class containing an element of B_n by \mathcal{C}_ρ^+ , then the other is $z \mathcal{C}_\rho^+$, and $\theta^{-1}(\mathcal{C}_\rho) = \mathcal{C}_\rho^+ \sqcup z \mathcal{C}_\rho^+$.

5.2. Simple modules of $\mathfrak{H}_{B_n}^\epsilon$. Propositions 3.3 and 4.3 imply that the simple $\mathfrak{H}_{B_n}^\epsilon$ -modules are parametrized by $\lambda \vdash n$, and that they are all of type M. We shall construct them explicitly, and then match them with the $\mathbb{C} B_n^-$ -modules B^λ via the super-equivalence in Proposition 3.3.

We adopt the convention that $c_{-i} = -c_i$ for $1 \leq i \leq n$. The algebra $\mathfrak{H}_{B_n}^\epsilon$ acts on the Clifford algebra \mathcal{Cl}_n by the formulas

$$c_i \cdot (c_{i_1} c_{i_2} \dots) = c_i c_{i_1} c_{i_2} \dots, \quad \sigma \cdot (c_{i_1} c_{i_2} \dots) = c_{\sigma(i_1)} c_{\sigma(i_2)} \dots,$$

for $\sigma \in B_n$ and all i . This $\mathfrak{H}_{B_n}^\epsilon$ -module \mathcal{Cl}_n is called the *basic spin module*.

Lemma 5.2. *The character value of the basic spin $\mathfrak{H}_{B_n}^\epsilon$ -module at an even split conjugacy class \mathcal{C}_ρ^+ is equal to $2^{\ell(\rho)}$ for $\rho \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$, and 0 elsewhere.*

Proof. Let $\sigma = \sigma_1 \dots \sigma_\ell \in B_n$ be a product of disjoint signed cycles of type ρ . The elements $c_I := \prod_{i \in I} c_i$ (which are defined up to a nonessential sign) for $I \subset \{1, \dots, n\}$ form a basis of the basic spin module \mathcal{Cl}_n . Observe that $\sigma c_I = c_I$ if I is a union of a subset of the supports $\text{supp}(\sigma_p)$ for $1 \leq p \leq \ell(\alpha)$; otherwise σc_I is equal to $\pm c_J$ for some $J \neq I$. Hence the lemma follows. \square

Let $\lambda \vdash n$. Via pullback of the canonical projection $B_n = \mathbb{Z}_2^n \rtimes S_n \rightarrow S_n$, the Specht module S^λ is endowed with a B_n -module structure. Then

$$(5.2) \quad K^\lambda := \mathcal{C}l_n \otimes S^\lambda$$

is naturally a module over $\mathfrak{H}_{B_n}^\epsilon = \mathcal{C}l_n \rtimes B_n$ (and hence also a module of the group $\tilde{\Gamma}_n$), where $\mathcal{C}l_n$ acts by left multiplication on the first tensor factor and B_n acts diagonally.

Denote by φ^λ the character of the module K^λ (of the group $\tilde{\Gamma}_n$). Note that the character value $\varphi^\lambda(x)$ is zero unless x is even split. There is a canonical bijection between the types $\rho = (\alpha, \beta, \emptyset, \emptyset)$ in $(\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$ such that $|\alpha| + |\beta| = n$ and partitions of n (by taking $\alpha \cup \beta$), and we shall denote the resulting partition by ρ again by abuse of notation. Note also that $x \in \mathcal{C}_\rho^+$ implies that $x^{-1} \in \mathcal{C}_\rho^+$. By Lemma 5.2 and the definition (5.2), we conclude that

$$(5.3) \quad \text{the character value } \varphi^\lambda(x) \text{ at } x \in \mathcal{C}_\rho^+ \text{ is } 2^{\ell(\rho)} \chi_\rho^\lambda,$$

where we recall χ_ρ^λ denotes the character value of S^λ at an element in S_n of cycle type ρ .

Thanks to the isomorphism $\Phi : \mathbb{C}\Gamma_n^- = \mathfrak{H}_{B_n}^\epsilon = \mathcal{C}l_V \rtimes B_n \rightarrow \mathcal{C}l_V \otimes \mathbb{C}B_n^-$ from Proposition 3.1 for $W = B_n$, the Morita super-equivalence in Proposition 3.3 applies. Further computation is necessary to determine which simple $\mathbb{C}B_n^-$ -module corresponds under the super-equivalence to the simple $\mathfrak{H}_{B_n}^\epsilon$ -module K^λ .

Lemma 5.3. *The $\mathfrak{H}_{B_n}^\epsilon$ -module K^λ corresponds to the $\mathbb{C}B_n^-$ -module B^λ under the bijection induced by \mathfrak{G} in Proposition 3.3.*

Proof. We shall compute the characters of the modules $U \otimes B^\lambda$ of $\mathfrak{H}_{B_n}^\epsilon$ (and hence of $\tilde{\Gamma}_n$) on an arbitrary even split class. A canonical representative for an even split class of Γ_n of type $(\rho_{++}, \rho_{+-}, \emptyset, \emptyset) \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$ can always be chosen inside the subgroup B_n as follows. We choose the canonical positive $(++)$ -cycle inside B_n which permutes a through $a+k$ to be $s_a \cdots s_{a+k-1}$, and the canonical negative $(+-)$ -cycle inside B_n which permutes those same elements to be $s_a \cdots s_{a+k-1} \tau_{a+k}$. A representative element for the conjugacy class of type $(\alpha, \beta, \emptyset, \emptyset)$ is chosen to be a product of such disjoint canonical cycles of the appropriate lengths, so that the supports of negative cycles are bigger than those of positive cycles.

Let $b_J \sigma$ be such a canonical representative element of the conjugacy class of type $(\rho_{++}, \rho_{+-}, \emptyset, \emptyset)$, and consider the images of its component cycles under Φ . Since there will be no cancellation between cycles, by Proposition 3.4 we may ignore terms containing nontrivial products of c_i .

Consider an odd positive $(k+1)$ -cycle $s_a \cdots s_{a+k-1}$ in $b_J \sigma$, for k even. Similar to the proof of Proposition 4.5, we compute

$$(5.4) \quad \begin{aligned} \Phi(s_a \cdots s_{a+k-1}) &= (-1)^{k+\frac{k}{2}} \beta_a t_a \cdots \beta_{a+k-1} t_{a+k-1} \\ &= (-1)^{k+\frac{k}{2}+\frac{k(k-1)}{2}} \beta_a \cdots \beta_{a+k-1} t_a \cdots t_{a+k-1} \\ &= 2^{-\frac{k}{2}} (-1)^{2k+\frac{k(k-1)}{2}} t_a \cdots t_{a+k-1} + (\text{terms involving } c_i) \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k}{2}} t_a \cdots t_{a+k-1} + (\text{terms involving } c_i). \end{aligned}$$

Now consider an even negative $(k+1)$ -cycle $s_a \cdots s_{a+k-1} \tau_{a+k}$ in $b_J \sigma$, for k odd. Recall τ_i is the generator of the i th copy of \mathbb{Z}_2 inside B_n , and b_i is defined in (4.2) for $1 \leq i \leq n$. By [KW2, Lemma 5.4], we have

$$(5.5) \quad \Phi(\tau_i) = (-1)^{n-i-\frac{1}{2}} c_i b_i, \quad \text{for } 1 \leq i \leq n.$$

As the positive cycles have supports in terms of smaller numbers than the negative cycles, we must have $n-a \equiv 1 \pmod{2}$. Using this parity condition, Table B for type B_n , (5.4) and (5.5), we compute

$$\begin{aligned} & \Phi(s_a \cdots s_{a+k-1} \tau_{a+k}) \\ &= (-1)^{k+\frac{k}{2}+\frac{k(k-1)}{2}+n-a-k-\frac{1}{2}} \beta_a \cdots \beta_{a+k-1} t_a \cdots t_{a+k-1} c_{a+k} b_{a+k} \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k(k-1)}{2}+n-a} t_a \cdots t_{a+k-1} b_{a+k} + (\text{terms involving } c_i) \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k^2-k}{2}+1} t_a \cdots t_{a+k-1} b_{a+k} + (\text{terms involving } c_i) \\ &= 2^{-\frac{k}{2}} (-1)^{\frac{k+1}{2}} t_a \cdots t_{a+k-1} b_{a+k} + (\text{terms involving } c_i). \end{aligned}$$

Here the last identity follows since $\frac{k^2-k}{2} + 1 \equiv \frac{k+1}{2} \pmod{2}$ whenever k is odd.

Multiplying the images of canonical cycles together, we obtain

$$\Phi(b_J \sigma) = 2^{-\frac{n-\ell(\rho_{++} \cup \rho_{+-})}{2}} (-1)^{\frac{n-\ell(\rho_{++})}{2}} \sigma + (\text{terms involving } c_i).$$

So the character value of $U \otimes B^\lambda$ on $b_J \sigma$ is

$$\begin{cases} 2^{\ell(\rho_{++} \cup \rho_{+-})} \chi_{\rho_{++} \cup \rho_{+-}}^\lambda & \text{if } n \text{ is even} \\ 2^{\ell(\rho_{++} \cup \rho_{+-})+1} \chi_{\rho_{++} \cup \rho_{+-}}^\lambda & \text{if } n \text{ is odd,} \end{cases}$$

which is equal to the character value of K^λ on $\Phi(b_J \sigma)$ when n is even, and twice that (since B^λ is of type Q) when n is odd; see (5.3). \square

Proposition 5.4. $\{K^\lambda \mid \lambda \vdash n\}$ is a complete list of pairwise inequivalent simple $\mathfrak{H}_{B_n}^\epsilon$ -modules, all of type M.

Proof. By Proposition 4.3 and Lemma 5.3, $\{K^\lambda \mid \lambda \vdash n\}$ are pairwise inequivalent simple $\mathfrak{H}_{B_n}^\epsilon$ -modules, all of type M. As we already know by Proposition 5.1 that the total number of simple modules is the number of partitions of n , the proof is completed. \square

5.3. The characteristic map for $\mathfrak{H}_{B_n}^\epsilon$. Let $R^-(\Gamma_n)$ be the Grothendieck group of the category of $\mathbb{C}\Gamma_n^-$ -modules, which can also be identified with the free abelian group with a basis made up of the irreducible $\mathbb{C}\Gamma_n^-$ -characters. Define

$$\check{R} = \bigoplus_{n=0}^{\infty} R^-(\Gamma_n),$$

where $R^-(\Gamma_0) = \mathbb{Z}$.

We shall define a ring structure on \check{R} as follows. For a $\mathbb{C}\Gamma_m^-$ -module M and a $\mathbb{C}\Gamma_n^-$ -module N , we define the product

$$[M] \cdot [N] = [\mathbb{C}\Gamma_{m+n}^- \otimes_{\mathbb{C}\Gamma_m^- \times \mathbb{C}\Gamma_n^-} (M \otimes N)],$$

and then extend by \mathbb{Z} -bilinearity. It follows from the properties of the induced characters that the multiplication on \check{R} is commutative and associative. Given $\mathbb{C}\Gamma_n^-$ -modules M, N , we define a bilinear form on \check{R} by letting

$$(5.6) \quad \langle M, N \rangle = \dim \operatorname{Hom}_{\mathbb{C}\Gamma_n^-}(M, N).$$

Now define the characteristic map $\operatorname{ch} : \check{R} \rightarrow \Lambda$ as the linear map

$$\operatorname{ch}(\phi) = \sum_{\mu \vdash n} z_\mu^{-1} 2^{-\ell(\mu)} \phi_\mu p_\mu, \quad \text{for } \phi \in R^-(\Gamma_n).$$

Lemma 5.5. *Let $\phi \in R^-(\Gamma_m), \psi \in R^-(\Gamma_n)$, and $\gamma \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$. Then*

$$(\phi \cdot \psi)(\gamma) = \sum_{\alpha, \beta} \frac{z_\gamma}{z_\alpha z_\beta} \phi(\alpha) \psi(\beta)$$

where the sum is taken over $\alpha, \beta \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$ such that $\gamma = \alpha \cup \beta$.

Proof. Let $g \in \tilde{\Gamma}_{m+n}$ be an element of type γ . By [Mac, I, Appendix B, (3.1)], the order of the centralizer in Γ_n of an element of type $\rho = (\alpha, \beta, \emptyset, \emptyset) \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$ is $z_\alpha z_\beta 4^{\ell(\alpha \cup \beta)}$. Now we compute

$$\begin{aligned} (\phi \cdot \psi)(\gamma) &= \frac{1}{|\tilde{\Gamma}_{m,n}|} \sum_{h \in \tilde{\Gamma}_{m+n}} (\phi \times \psi)(h^{-1}gh) \\ &= \frac{|\tilde{\Gamma}_{n+m}|}{|\tilde{\Gamma}_{m,n}| |\mathcal{C}_\gamma^+|} \sum_{w \in \mathcal{C}_\gamma^+} (\phi \times \psi)(w) \\ &= \frac{2^{2\ell(\gamma)} z_\gamma}{m! n! 2^{2m+2n}} \sum_{\alpha \cup \beta = \gamma} \phi(\alpha) \psi(\beta) |\mathcal{C}_\alpha^+| |\mathcal{C}_\beta^+| \\ &= \frac{2^{2\ell(\gamma)} z_\gamma}{m! n! 2^{2m+2n}} \sum_{\alpha \cup \beta = \gamma} \phi(\alpha) \psi(\beta) \frac{2^{2m} m!}{z_\alpha 2^{2\ell(\alpha)}} \frac{2^{2n} n!}{z_\beta 2^{2\ell(\beta)}} \\ &= \sum_{\alpha, \beta} \frac{z_\gamma}{z_\alpha z_\beta} \phi(\alpha) \psi(\beta). \end{aligned}$$

The lemma is proved. \square

Theorem 5.6. *The characteristic map $\operatorname{ch} : \check{R} \rightarrow \Lambda$ is an isometric isomorphism of graded algebras, which sends $[K^\lambda]$ to s_λ for all λ .*

Proof. Recall that the character φ^λ of the irreducible module K^λ is $2^{\ell(\alpha \cup \beta)} \chi_{\alpha \cup \beta}^\lambda$ on elements of type $(\alpha, \beta, \emptyset, \emptyset) \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$, and 0 otherwise. Thus

$$\operatorname{ch}(\varphi^\lambda) = \sum_{(\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})} z_\alpha^{-1} z_\beta^{-1} 2^{-\ell(\alpha \cup \beta)} 2^{\ell(\alpha \cup \beta)} \chi_{\alpha \cup \beta}^\lambda p_{\alpha \cup \beta} = s_\lambda.$$

This map sends an orthonormal basis for \check{R} to an orthonormal basis of Λ , so it is an isometry.

Now we compute the image of a product under the characteristic map. Let ϕ, ψ be as in the previous lemma. Then

$$\begin{aligned} \text{ch}(\phi \cdot \psi) &= \sum_{\gamma \vdash m+n} z_\gamma^{-1} 2^{\ell(\gamma)} (\phi \cdot \psi)(\gamma) p_\gamma \\ &= \sum_{\gamma} \sum_{\alpha, \beta: \alpha \cup \beta = \gamma} z_\gamma^{-1} \frac{z_\gamma}{z_\alpha z_\beta} \phi(\alpha) \psi(\beta) 2^{\ell(\gamma)} p_\gamma = \text{ch}(\phi) \text{ch}(\psi), \end{aligned}$$

so ch is also an algebra isomorphism. \square

5.4. Spin fake degrees for $\mathfrak{H}_{B_n}^c$. Let x, y , and z be three (possibly infinite) sets of independent indeterminates. The super Schur functions and the super Cauchy identity are standard, and we refer to [WW2, Section 5.3] for more details. For a partition λ , the *super Schur function* hs_λ is defined to be

$$hs_\lambda(x; y) := \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda' / \mu'}(y).$$

We have the following super Cauchy identity:

$$(5.7) \quad \frac{\prod_{j,k} (1 + y_j z_k)}{\prod_{i,k} (1 - x_i z_k)} = \sum_{\lambda \in \mathcal{P}} hs_\lambda(x; y) s_\lambda(z).$$

Let a, b be indeterminates. The formula (*) in [Mac, I.3.3] was interpreted in [WW2, (5.13)] as a specialization of $hs_\lambda(x; y)$, by letting $x = aq^\bullet = (a, aq, aq^2, \dots)$ and $y = bq^\bullet$:

$$(5.8) \quad hs_\lambda(aq^\bullet; bq^\bullet) = q^{n(\lambda)} \prod_{\square \in \lambda} \frac{a + bq^{c_\square}}{1 - q^{h_\square}}.$$

The following theorem was used in the proof of Theorem 4.9 earlier.

Theorem 5.7. *The graded multiplicity of the irreducible $\mathfrak{H}_{B_n}^c$ -module K^λ in the $\mathfrak{H}_{B_n}^c$ -module $\mathcal{Cl}_V \otimes S^*V$ is*

$$H_{B_n}(\lambda, t) = hs_\lambda(t^{2\bullet}; t^{2\bullet+1}) = t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_\square+1}}{1 - t^{2h_\square}}.$$

Proof. The second identity follows by (5.8) with $a = 1, b = t$ and $q = t^2$. So it remains to prove the first identity.

Let us compute the image of $\mathcal{Cl}_V \otimes S_t V$ under the characteristic map. The character of the basic spin $\mathbb{C}\Gamma_n^-$ -module \mathcal{Cl}_V on any representative in the conjugacy class of type $(\alpha, \beta, \emptyset, \emptyset) \in (\mathcal{OP}, \mathcal{EP}, \emptyset, \emptyset)$ is $2^{\ell(\alpha \cup \beta)}$; we shall choose the representative to be the canonical element $b_J \sigma \in B_n$ (with $\sigma \in S_n$) of type $(\alpha, \beta) \in (\mathcal{OP}, \mathcal{EP})$ as in the proof of Lemma 5.3.

Now we compute the character value of $b_J \sigma$ on the B_n -module S^*V . We shall denote by $\ell_1 = \ell(\alpha)$, $\ell_2 = \ell(\beta)$, $\ell = \ell(\alpha) + \ell(\beta)$. We write

$$\sigma = (1, \dots, \alpha_1)(\alpha_1 + 1, \dots, \alpha_1 + \alpha_2) \cdots (|\alpha| + 1, \dots, |\alpha| + \beta_1) \cdots (n - \beta_{\ell_2} + 1, \dots, n).$$

Thus σ will permute the monomial basis of S^*V , fixing only those monomials

$$\underline{x}^a := (x_1 x_2 \cdots x_{\alpha_1})^{a_1} (x_{\alpha_1+1} \cdots x_{\alpha_1+\alpha_2})^{a_2} \cdots (x_{n-\beta_{\ell_2}+1} \cdots x_n)^{a_\ell}$$

for nonnegative integers a_1, \dots, a_ℓ . Note that

$$b_J \sigma(\underline{x}^{\underline{a}}) = (-1)^{a_{\ell_1+1} + \dots + a_\ell} \underline{x}^{\underline{a}}.$$

This implies that the character value of $S_t V$ on $b_J \sigma$ is

$$\begin{aligned} \text{tr}(b_J \sigma)|_{S_t V} &= \sum_{a_1, \dots, a_\ell \geq 0} (-1)^{a_{\ell_1+1} + \dots + a_\ell} t^{\sum_i a_i \alpha_i + \sum_j a_{\ell_1+j} \beta_j} \\ &= \frac{1}{(1-t^{\alpha_1}) \dots (1-t^{\alpha_{\ell_1}})(1+t^{\beta_1}) \dots (1+t^{\beta_{\ell_2}})} \\ &= \frac{1}{(1+(-t)^{\rho_1}) \dots (1+(-t)^{\rho_\ell})}, \end{aligned}$$

where we have switched notation in the last equation using $\rho = \alpha \cup \beta = (\rho_1, \rho_2, \dots, \rho_\ell)$. Then the character value of $\mathcal{C}l_V \otimes S_t V$ on $b_J \sigma$ is

$$(5.9) \quad \text{tr}(b_J \sigma)|_{\mathcal{C}l_V \otimes S_t V} = \frac{2^{\ell(\rho)}}{(1+(-t)^{\rho_1}) \dots (1+(-t)^{\rho_\ell})}.$$

Given a power series $f(u)$ in a variable u , we denote by $[u^n]f(u)$ the coefficient of u^n in the series expansion of $f(u)$. Applying the characteristic map to $\mathcal{C}l_n \otimes S_t V$ with the help of (5.9), we compute

$$\begin{aligned} \text{ch}(\mathcal{C}l_n \otimes S_t V) &= \sum_{\rho \vdash n} z_\rho^{-1} 2^{-\ell(\rho)} \frac{2^{\ell(\rho)}}{(1+(-t)^{\rho_1}) \dots (1+(-t)^{\rho_\ell})} p_\rho \\ &= [u^n] \sum_{\rho \in \mathcal{P}} z_\rho^{-1} p_\rho \frac{u^{|\rho|}}{(1+(-t)^{\rho_1}) \dots (1+(-t)^{\rho_\ell})} \\ &= [u^n] \prod_{i,j} \left(\frac{1}{1-x_i u(-t)^j} \right)^{(-1)^j} \\ &= \sum_{\lambda \vdash n} h s_\lambda(t^{2\bullet}; t^{2\bullet+1}) s_\lambda(x). \end{aligned}$$

The last equation used the super Cauchy identity (5.7). On the other hand, since each K^λ is simple of type \mathbb{M} by Proposition 5.4, we have

$$\text{ch}(\mathcal{C}l_n \otimes S_t V) = \sum_{\lambda \vdash n} H_{B_n}(\lambda, t) s_\lambda(x).$$

The first identity in the theorem now follows from comparing the above two expressions for $\text{ch}(\mathcal{C}l_n \otimes S_t V)$, and the linear independence of s_λ 's. \square

The following is equivalent to Theorem 5.7 by Lemma 3.11 and using the fact that the degrees of B_n are $2, 4, \dots, 2n$.

Theorem 5.8. *The spin fake degree of the irreducible $\mathfrak{H}_{B_n}^c$ -module K^λ is*

$$P_{B_n}(\lambda, t) = t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1+t^{2c_\square+1}}{1-t^{2h_\square}} (1-t^2)(1-t^4) \dots (1-t^{2n}).$$

6. THE SPIN FAKE DEGREES OF TYPE D_n (FOR n ODD)

Let n be odd throughout this section.

6.1. Structure of the algebra $\mathbb{C}D_n^-$ for n odd. Recall from Table A the definition of $\mathbb{C}D_n^-$ via generators t_1, \dots, t_n and defining relations. Denote by $\mathcal{C}l_n^0 \subset \mathcal{C}l_n$ the even subalgebra of the Clifford superalgebra $\mathcal{C}l_n$. The algebra $\mathcal{C}l_n^0$ is abstractly a Clifford algebra in $n - 1$ generators $c_i c_n$ for $1 \leq i \leq n - 1$. Define

$$\zeta := (-1)^{\frac{n(n-1)}{2}} c_1 c_2 \cdots c_n \in \mathcal{C}l_n.$$

Note that $\zeta \notin \mathcal{C}l_n^0$.

Lemma 6.1. *For n odd, the element ζ commutes with every c_i . Moreover, $\zeta^2 = 1$.*

Proof. Follows by a direct computation. □

The tensor algebra $\mathcal{C}l_n^0 \otimes \mathbb{C}S_n$ carries a superalgebra structure by letting each $c_i c_n$ be even and each s_i odd, for $1 \leq i \leq n - 1$. In particular, $\mathcal{C}l_n^0$ is a purely even algebra, and it follows by (3.1) that $c_i c_n$ commutes with s_j for all possible i, j .

Theorem 6.2. *For n odd, there exists an isomorphism of superalgebras*

$$\begin{aligned} \phi^D : \mathbb{C}D_n^- &\xrightarrow{\cong} \mathcal{C}l_n^0 \otimes \mathbb{C}S_n, \\ t_i &\mapsto \begin{cases} \frac{1}{\sqrt{2}} \zeta (c_i - c_{i+1}) s_i, & \text{for } 1 \leq i \leq n - 1, \\ \frac{1}{\sqrt{2}} \zeta (c_{n-1} + c_n) s_{n-1}, & \text{for } i = n. \end{cases} \end{aligned}$$

(We emphasize here that each s_i is odd.)

Proof. First note that $\phi^D(t_i)$ for each i is indeed in $\mathcal{C}l_n^0 \otimes \mathbb{C}S_n$, since n is odd.

Recall from Theorem 4.1 the superalgebra isomorphism $\phi^B : \mathbb{C}B_n^- \xrightarrow{\cong} \mathcal{C}l_n \otimes \mathbb{C}S_n$. The images of t_i for $1 \leq i \leq n - 1$ under ϕ^D and ϕ^B differ exactly by a factor ζ . All the relations for $\mathbb{C}D_n^-$ which do not involve t_n in Table B are identical for types B and D and they all involve even numbers of these t_i 's, hence they are preserved by ϕ^D because $\zeta^2 = 1$ and ϕ^B is a homomorphism. In addition, it is straightforward to check by definition and Lemma 6.1 the remaining relations:

$$\begin{aligned} (\phi^D(t_i) \phi^D(t_n))^2 &= -1, \quad \text{for } i \neq n - 2, n, \\ \phi^D(t_n)^2 &= 1, \quad (\phi^D(t_n) \phi^D(t_{n-2}))^3 = 1. \end{aligned}$$

For example, we compute

$$(\phi^D(t_n) \phi^D(t_{n-2}))^3 = -\frac{1}{8} \zeta^6 ((c_{n-1} + c_n)(c_{n-2} - c_{n-1}))^3 (s_{n-1} s_{n-2})^3 = 1.$$

Hence, ϕ^D is an algebra homomorphism. Also, ϕ^D preserves the superalgebra structures since $\phi^D(t_i)$ and t_i for each i are odd.

To show that $\phi^D : \mathbb{C}D_n^- \rightarrow \mathcal{C}l_n^0 \otimes \mathbb{C}S_n$ is an isomorphism, it remains to verify the surjectivity as both algebras have the same dimension. Equivalently, it suffices to check

that the generators $c_i c_n$ and s_i , for $1 \leq i \leq n-1$, of $\mathcal{Cl}_n^0 \otimes \mathbb{C}S_n$ lie in the image of ϕ^D . To that end, a direct computation shows that

$$(6.1) \quad \begin{aligned} \phi^D(t_{n-1})\phi^D(t_n) &= c_{n-1}c_n, \\ \phi^D(t_i)c_{i+1}c_n\phi^D(t_i) &= c_i c_n, \quad \text{for } 1 \leq i \leq n-2. \end{aligned}$$

Inductively we conclude by (6.1) that all $c_i c_n$, and hence \mathcal{Cl}_n^0 , lie in the image of ϕ^D . Now we can choose $x_i \in \mathbb{C}D_n^-$ such that $\phi^D(x_i) = \sqrt{2}(\zeta(c_i - c_{i+1}))^{-1} \in \mathcal{Cl}_n^0$, for $1 \leq i \leq n-1$. Then $\phi^D(x_i t_i) = \phi^D(x_i)\phi^D(t_i) = s_i$. Thus the homomorphism ϕ^D is surjective. The theorem is proved. \square

There is a natural inclusion [KW3, 4.1]

$$(6.2) \quad \iota : \mathbb{C}D_n^- \hookrightarrow \mathbb{C}B_n^-,$$

which sends $t_i^D \mapsto t_i^B$ ($i \leq n-1$) and $t_n^D \mapsto -t_n^B t_{n-1}^B t_n^B$, if we use superscripts to indicate the types of Weyl groups. By Lemma 6.1, the superalgebra $\langle \zeta \rangle$ generated by ζ is isomorphic to \mathcal{Cl}_1 . Recall $|A|$ denotes the underlying algebra for a superalgebra A .

Proposition 6.3. *We have an isomorphism of algebras:*

$$\begin{aligned} |\mathbb{C}D_n^-| \times |\langle \zeta \rangle| &\xrightarrow{\cong} |\mathbb{C}B_n^-|, \\ (x, \zeta^a) &\mapsto \iota(x)\zeta^a, \quad \text{for } x \in |\mathbb{C}D_n^-|, a = 0, 1. \end{aligned}$$

Note that we cannot claim to have a superalgebra isomorphism $\mathbb{C}D_n^- \times \langle \zeta \rangle \xrightarrow{\cong} \mathbb{C}B_n^-$, since the odd element ζ commutes but does not super-commute with $\mathbb{C}D_n^-$.

Proof. If we identify $\mathbb{C}B_n^- \equiv \mathcal{Cl}_n \rtimes_- \mathbb{C}S_n^-$ as in Theorem 4.1, we can naturally identify the subalgebra $\iota(\mathbb{C}D_n^-) \equiv \mathcal{Cl}_n^0 \rtimes_- \mathbb{C}S_n^-$. Putting Theorem 4.1, Lemma 6.1 and Theorem 6.2 together, we have the following commutative diagrams, where the homomorphism j extends the natural inclusion $\mathcal{Cl}_n^0 \hookrightarrow \mathcal{Cl}_n$ and sends $s_i \mapsto \zeta s_i$ for each i :

$$(6.3) \quad \begin{array}{ccc} \mathcal{Cl}_n^0 \rtimes_- \mathbb{C}S_n^- & \xrightarrow{=} & \mathbb{C}D_n^- & |\mathbb{C}D_n^-| & \xrightarrow{\phi^D} & |\mathcal{Cl}_n^0 \otimes \mathbb{C}S_n| \\ \downarrow & & \downarrow \iota & \downarrow \iota & & \downarrow j \\ \mathcal{Cl}_n \rtimes_- \mathbb{C}S_n^- & \xrightarrow{=} & \mathbb{C}B_n^- & |\mathbb{C}B_n^-| & \xrightarrow{\phi^B} & |\mathcal{Cl}_n \otimes \mathbb{C}S_n| \end{array}$$

It follows that $\zeta \notin \iota(\mathbb{C}D_n^-)$ and that ζ commutes with $\iota(\mathbb{C}D_n^-)$. The proposition is proved. \square

Recall S^λ denotes the Specht module of S_n . Denote by U^0 the unique simple \mathcal{Cl}_n^0 -module. Theorem 6.2 implies immediately the following classification of simple $|\mathbb{C}D_n^-|$ -modules, which was obtained by Read [Re2, Theorem 7.2] using a completely different construction of these simple modules.

Corollary 6.4. *Let n be odd. A complete list of pairwise inequivalent simple $|\mathbb{C}D_n^-|$ -modules is $\{U^0 \otimes S^\lambda \mid \lambda \vdash n\}$.*

6.2. Split classes for n odd.

Lemma 6.5. *Let n be odd.*

- (1) *The split conjugacy classes of D_n are the classes of the following types:*
 - (a) $(\rho_+, \rho_-) \in (\mathcal{OP}, \mathcal{EP})$, with $\ell(\rho_-)$ even;
 - (b) $(\rho_+, \rho_-) \in (\emptyset, \mathcal{P})$, with $\ell(\rho_-)$ even.
- (2) *The split classes of type $(\rho_+, \rho_-) \in (\mathcal{OP}, \mathcal{EP})$ are even while those of type $(\rho_+, \rho_-) \in (\emptyset, \mathcal{P})$ are odd.*

Proof. (1) is [Re2, Lemmas 6.4, 7.1]. (2) follows by counting the number of generators in a representative element of each conjugacy class as given in [Re2], and noting that each generator t_i of $\mathbb{C}D_n^-$ is odd. \square

Remark 6.6. Note that $\ell(\rho_+)$ must be odd in the case of Lemma 6.5(1a). Hence the types of the split classes of D_n are in natural bijection with the partitions of n by sending $(\rho_+, \rho_-) \mapsto \rho_+ \cup \rho_-$, so that the classes in (1a) (respectively, (1b)) correspond to partitions of n of odd lengths (respectively, even lengths).

6.3. Counting the simples. Denote by \mathcal{P}_n the set of partitions of n , $\mathcal{P}_n^{\text{od}}$ the set of partitions of n of odd length, $\mathcal{P}_n^{\text{ev}}$ the set of partitions of n of even length, $\mathcal{P}_n^{\text{sym}}$ the set of (conjugate-)symmetric partitions of n , and \mathcal{SOP}_n the set of strict odd partitions of n .

Lemma 6.7. *Let n be odd.*

- (1) *The number of simple $\mathbb{C}D_n^-$ -modules of type M is equal to either*
 - (a) $|\{\lambda \vdash n : n - \ell(\lambda) \text{ even}\}| - |\{\lambda \vdash n : n - \ell(\lambda) \text{ odd}\}|$;
 - (b) $|\mathcal{SOP}_n|$;
 - (c) $|\mathcal{P}_n^{\text{sym}}|$.
- (2) *The number of simple $\mathbb{C}D_n^-$ -modules of type Q is equal to either*
 - (a) $|\{\lambda \vdash n : n - \ell(\lambda) \text{ odd}\}|$;
 - (b) $|\{\{\lambda, \lambda'\} : \lambda' \neq \lambda \in \mathcal{P}_n\}|$;
 - (c) $\frac{1}{2}(|\mathcal{P}_n| - |\mathcal{P}_n^{\text{sym}}|)$.

Proof. Denote by m (and respectively, q) the number of simple $\mathbb{C}D_n^-$ -modules of type M (and respectively, type Q). Then the number of simple $|\mathbb{C}D_n^-|$ -modules is $m + 2q$, which should be equal to $|\mathcal{P}_n|$ by counting the split classes in Lemma 6.5 and applying Wedderburn's Theorem 2.2 and Remark 2.4. From this we easily see that (1) is consistent with (2), and so it suffices to prove (1).

According to Proposition 2.5 and Lemma 6.5, m is given by (1a). It is a classical fact that $(1a) = (1b) = (1c)$ (which is valid actually for any n). The equality $(1b) = (1c)$ can be shown by an easy bijection, while the equality $(1a) = (1b)$ can be established via a generating function identity: $\prod_{i \geq 1} (1 + (-x)^i)^{-1} = \prod_{i \geq 1} (1 + x^{2i-1})$. \square

6.4. Classification of simple $\mathbb{C}S_n$ -(super)modules. Recall that χ_μ^λ is the character of the Specht module S^λ on the conjugacy class of S_n of cycle type μ .

Lemma 6.8. *Let n be odd. Then $\chi_\mu^\lambda = 0$ for all $\mu \vdash n$ of even length if and only if $\lambda = \lambda'$.*

Proof. Note that

$$\chi^{\lambda'} = \chi^\lambda \otimes \text{sgn}, \quad \text{sgn}_\mu = (-1)^{n-\ell(\mu)} = -(-1)^{\ell(\mu)},$$

which we shall use repeatedly below.

(\Leftarrow) Assume $\lambda = \lambda'$. If $\ell(\mu)$ is even, then $\chi_\mu^\lambda = \chi_\mu^{\lambda'} = \chi_\mu^\lambda \cdot \text{sgn}_\mu = -\chi_\mu^\lambda$, so $\chi_\mu^\lambda = 0$.

(\Rightarrow) Let $\nu, \mu \vdash n$ with $\ell(\mu)$ even, and $\ell(\nu)$ odd. Then $\chi_\mu^{\lambda'} = -\chi_\mu^\lambda = 0 = \chi_\mu^\lambda$. Also, $\chi_\nu^{\lambda'} = \chi_\nu^\lambda \cdot \text{sgn}_\nu = (-1)^{n-\ell(\nu)} \chi_\nu^\lambda = \chi_\nu^\lambda$. Hence $\chi^\lambda = \chi^{\lambda'}$ and so $\lambda = \lambda'$. \square

Proposition 6.9. *Let $\mathbb{C}S_n$ be endowed with the superalgebra structure with each s_i odd. Then a complete list of pairwise inequivalent simple $\mathbb{C}S_n$ -modules consists of:*

- (1) S^λ of type \mathbf{M} , for $\lambda \vdash n$ with $\lambda = \lambda'$.
- (2) $S^{\{\lambda, \lambda'\}} := S^\lambda \oplus S^{\lambda'}$ of type \mathbf{Q} , for pairs $\{\lambda, \lambda'\}$ with $\lambda \vdash n$ and $\lambda \neq \lambda'$.

Proof. Given a finite-dimension semisimple \mathbb{C} -superalgebra A , one defines an involution α on A by letting $\alpha(a) = (-1)^{|a|}a$ for homogeneous $a \in A$. Given any $|A|$ -module N , one obtains another $|A|$ -module N' with the same underlying vector space as N but with an action twisted by α . It is shown in [Joz1, Proposition 2.17] that

- (i) If N is a simple $|A|$ -module but not an A -module, then $N \not\cong N'$ and $N \oplus N'$ can be endowed with a simple A -module structure of type \mathbf{Q} .
- (ii) If a simple $|A|$ -module N can be lifted to a simple A -module (which must be of type \mathbf{M}), then $N \cong N'$.

In our setting with $A = \mathbb{C}S_n$, the simple $\mathbb{C}S_n$ -modules are S^λ , the involution is given by $\alpha(\sigma) = (-1)^{l(\sigma)}\sigma$ for $\sigma \in S_n$, and the twisted module N' is isomorphic to $N \otimes \text{sgn}$. Now (1) follows from (ii) above and the fact that $S^\lambda \otimes \text{sgn} \cong S^{\lambda'}$. By Lemma 6.7, we have obtained all simple $\mathbb{C}S_n$ -modules of type \mathbf{M} . Then S^λ for $\lambda \neq \lambda'$ must pair with $S^{\lambda'}$ to give rise to simple $\mathbb{C}S_n$ -modules of type \mathbf{Q} , by applying (i) above. That S^λ for $\lambda \neq \lambda'$ is not a $\mathbb{C}S_n$ -module also follows from Lemma 6.8, which says S^λ has non-vanishing character value on odd conjugacy classes. \square

6.5. Simple $\mathbb{C}D_n^-$ -modules for n odd. Recall U^0 denotes the unique simple $\mathcal{C}l_n^0$ -module. We introduce the following notation for n odd:

$$(6.4) \quad D^{\{\lambda, \lambda'\}} := \begin{cases} U^0 \otimes S^\lambda & \text{if } \lambda = \lambda', \\ U^0 \otimes (S^\lambda \oplus S^{\lambda'}) & \text{if } \lambda \neq \lambda'. \end{cases}$$

By definition $D^{\{\lambda, \lambda'\}}$ only depends on the unordered pair $\{\lambda, \lambda'\}$. Via the isomorphism ϕ^D from Theorem 6.2, $D^{\{\lambda, \lambda'\}}$ is a $\mathbb{C}D_n^-$ -module.

Proposition 6.10. *Let n be odd. Then $\{D^{\{\lambda, \lambda'\}} \mid \lambda \vdash n\}$ forms a complete set of pairwise inequivalent simple $\mathbb{C}D_n^-$ -modules. Moreover, $D^{\{\lambda, \lambda'\}}$ is of type \mathbf{M} if $\lambda = \lambda'$, and of type \mathbf{Q} otherwise.*

Proof. Follows from Theorem 6.2 and Proposition 6.9, by noting that $\mathcal{C}l_n^0$ is purely even and the unique simple $\mathcal{C}l_n^0$ -module U^0 is of type \mathbf{M} . \square

Recall the irreducible $\mathbb{C}B_n^-$ -modules B^λ for $\lambda \vdash n$ from Proposition 4.3, which are all of type \mathbf{Q} since n is odd. Recall from (6.2) the inclusion $\mathbb{C}D_n^- \hookrightarrow \mathbb{C}B_n^-$.

Proposition 6.11. *Let n be odd. Then*

- (1) *As a $|\mathbb{C}D_n^-|$ -module, B^λ is a sum of two simple modules, i.e., $B^\lambda \cong U^0 \otimes S^\lambda \oplus U^0 \otimes S^{\lambda'}$, for $\lambda \vdash n$.*
- (2) *$B^\lambda|_{\mathbb{C}D_n^-} \cong B^{\lambda'}|_{\mathbb{C}D_n^-} \cong D^{\{\lambda, \lambda'\}}$, for $\lambda \vdash n$ with $\lambda \neq \lambda'$.*
- (3) *$B^\lambda|_{\mathbb{C}D_n^-} \cong (D^{\{\lambda, \lambda'\}})^{\oplus 2}$, for $\lambda \vdash n$ with $\lambda = \lambda'$.*

Proof. By construction, $B^\lambda \cong U \otimes S^\lambda$ via the isomorphism ϕ^B , where U is the simple Cl_n -module of type \mathbb{Q} . U decomposes into a sum of two copies of the Cl_n^0 -module U^0 , on which ζ acts by ± 1 respectively. By the (second) commutative diagram in (6.3), the action of $s_i \in \mathbb{C}S_n \subseteq \phi^D(\mathbb{C}D_n^-)$ on $U \otimes S^\lambda$ is twisted by the action of ζ , giving rise to $U^0 \otimes S^\lambda \oplus U^0 \otimes S^{\lambda'}$, whence (1).

Assume $\lambda \neq \lambda'$. Then $B^\lambda|_{\mathbb{C}D_n^-}$ must be isomorphic to the simple $\mathbb{C}D_n^-$ -module $D^{\{\lambda, \lambda'\}}$ by applying (1) and Proposition 6.10, whence (2).

Part (3) follows by applying (1) and Proposition 6.10 again. \square

6.6. Spin fake degrees of D_n for n odd. Recall \mathcal{B}_{D_n} is the basic spin $\mathbb{C}D_n^-$ -module. We would like to compute

$$(6.5) \quad \begin{aligned} H_{D_n}^-(\lambda\lambda', t) &:= \sum_k \dim \operatorname{Hom}_{\mathbb{C}D_n^-}(D^{\{\lambda, \lambda'\}}, \mathcal{B}_{D_n} \otimes S^k V) t^k, \\ P_{D_n}^-(\lambda\lambda', t) &:= \sum_k \dim \operatorname{Hom}_{\mathbb{C}D_n^-}(D^{\{\lambda, \lambda'\}}, \mathcal{B}_{D_n} \otimes (S^k V)_{D_n}) t^k. \end{aligned}$$

Proposition 6.11 allows us to reduce the computations to the type B_n case.

Theorem 6.12. *Let n be odd. Then*

$$H_{D_n}^-(\lambda\lambda', t) = \begin{cases} 2t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_\square + 1}}{1 - t^{2h_\square}}, & \text{if } \lambda = \lambda', \\ \frac{2t^{2n(\lambda)}}{\prod_{\square \in \lambda} (1 - t^{2h_\square})} \left(\prod_{\square \in \lambda} (1 + t^{2c_\square + 1}) + \prod_{\square \in \lambda} (t^{2c_\square} + t) \right), & \text{if } \lambda \neq \lambda'. \end{cases}$$

Proof. Note that $\mathcal{B}_{D_n} \cong \mathcal{B}_{B_n}|_{\mathbb{C}D_n^-}$. So we have

$$\mathcal{B}_{D_n} \otimes S^* V \cong (\mathcal{B}_{B_n} \otimes S^* V)|_{\mathbb{C}D_n^-} = \bigoplus_{\lambda \vdash n} \frac{1}{2} H_{B_n}^-(\lambda, t) B^\lambda|_{\mathbb{C}D_n^-};$$

here the factor $\frac{1}{2}$ arises since by Proposition 4.3 all $\mathbb{C}B_n^-$ -modules B^λ are type \mathbb{Q} when n is odd. Also, by definition

$$\mathcal{B}_{D_n} \otimes S^* V = \bigoplus_{\lambda = \lambda'} H_{D_n}^-(\lambda\lambda', t) D^{\{\lambda, \lambda'\}} \oplus \bigoplus_{\{\lambda, \lambda' | \lambda \neq \lambda'\}} \frac{1}{2} H_{D_n}^-(\lambda\lambda', t) D^{\{\lambda, \lambda'\}}.$$

By (6.4), Proposition 6.11 and a comparison of the above two expansions for $\mathcal{B}_{D_n} \otimes S^* V$, we have

$$(6.6) \quad H_{D_n}^-(\lambda\lambda', t) = \begin{cases} H_{B_n}^-(\lambda, t), & \text{if } \lambda = \lambda', \\ H_{B_n}^-(\lambda, t) + H_{B_n}^-(\lambda', t), & \text{if } \lambda \neq \lambda'. \end{cases}$$

Recall the formula for $H_{B_n}^-(\lambda, t)$ in Theorem 4.9. We can rewrite

$$(6.7) \quad H_{B_n}^-(\lambda', t) = 2^{p(n)} t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{t^{2c_\square} + t}{1 - t^{2h_\square}},$$

with $p(n) = 1$, using the following identities:

$$t^{2n(\lambda')} \prod_{\square \in \lambda'} (1 + t^{2c_\square+1}) = \prod_{(i,j) \in \lambda'} (t^{2(i-1)} + t^{2(j-1)+1}) = t^{2n(\lambda)} \prod_{\square \in \lambda} (t^{2c_\square} + t).$$

Now the theorem follows by applying to (6.6) the formulas in Theorem 4.9 and (6.7). \square

The following is equivalent to Theorem 6.12 by Lemma 3.11 and using the well-known fact that the degrees of D_n are $2, 4, \dots, 2n - 2, n$.

Theorem 6.13. *Let n be odd and $\lambda \vdash n$. Then the spin fake degree $P_{D_n}^-(\lambda\lambda', t)$ is*

$$\begin{cases} 2t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_\square+1}}{1 - t^{2h_\square}} \prod_{i=1}^{n-1} (1 - t^{2i})(1 - t^n), & \text{if } \lambda = \lambda', \\ \frac{2t^{2n(\lambda)}}{\prod_{\square \in \lambda} (1 - t^{2h_\square})} \left(\prod_{\square \in \lambda} (1 + t^{2c_\square+1}) + \prod_{\square \in \lambda} (t^{2c_\square} + t) \right) \prod_{i=1}^{n-1} (1 - t^{2i})(1 - t^n), & \text{if } \lambda \neq \lambda'. \end{cases}$$

Proposition 6.14. *The following palindromicity holds for spin fake degrees of D_n , for all $\lambda \vdash n$:*

$$P_{D_n}^-(\lambda\lambda', t) = t^{n(n-1)} P_{D_n}^-(\lambda\lambda', t^{-1}).$$

Proof. Using Lemma 3.11 and (6.6) while keeping in mind the degrees for B_n and D_n , we relate $P_{D_n}^-$ to $P_{B_n}^-$ as follows:

$$P_{D_n}^-(\lambda\lambda', t) = \begin{cases} P_{B_n}^-(\lambda, t) \frac{(1 - t^n)}{(1 - t^{2n})}, & \text{if } \lambda = \lambda', \\ (P_{B_n}^-(\lambda, t) + P_{B_n}^-(\lambda', t)) \frac{(1 - t^n)}{(1 - t^{2n})}, & \text{if } \lambda \neq \lambda'. \end{cases}$$

Since $P_{B_n}^-(\lambda, t)$ and $P_{B_n}^-(\lambda', t)$ are palindromic with the same shift number n^2 , their sum will be as well. It remains palindromic upon multiplication by $\frac{(1-t^n)}{(1-t^{2n})}$, with a new shift number $(n^2 - n)$. \square

6.7. Hecke-Clifford algebra $\mathfrak{H}_{D_n}^\epsilon$ for n odd. Recall from Proposition 5.4 that the simple $\mathfrak{H}_{B_n}^\epsilon$ -modules are $K^\lambda = \mathcal{C}l_n \otimes S^\lambda$ defined in (5.2) for $\lambda \vdash n$ and they are all of type \mathbb{M} .

Lemma 6.15. *Let n be odd. If $\lambda = \lambda'$, then $K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}$ is a type Q simple $\mathfrak{H}_{D_n}^\epsilon$ -module. Otherwise, if $\lambda \neq \lambda'$, then $K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}$ is a type M simple $\mathfrak{H}_{D_n}^\epsilon$ -module.*

Proof. This follows from Propositions 3.1, 3.3, 6.10 and 6.11. \square

Denote

$$H_{D_n}(\lambda\lambda', t) := \sum_k \dim \operatorname{Hom}_{\mathfrak{H}_{D_n}^\epsilon}(K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}, \mathcal{C}l_V \otimes S^k V) t^k,$$

$$P_{D_n}(\lambda\lambda', t) := \sum_k \dim \operatorname{Hom}_{\mathfrak{H}_{D_n}^\epsilon}(K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}, \mathcal{C}l_V \otimes (S^k V)_{D_n}) t^k.$$

Recall the formulas for $H_{D_n}^-(\lambda\lambda', t)$ from Theorem 6.12, and the formulas for $P_{D_n}^-(\lambda\lambda', t)$ from Theorem 6.13. The following proposition allows us to compute closed formulas for $H_{D_n}(\lambda\lambda', t)$ and $P_{D_n}(\lambda\lambda', t)$.

Proposition 6.16. *Let n be odd. If $\lambda = \lambda'$, then*

$$H_{D_n}(\lambda\lambda', t) = H_{D_n}^-(\lambda\lambda', t), \quad P_{D_n}(\lambda\lambda', t) = P_{D_n}^-(\lambda\lambda', t).$$

If $\lambda \neq \lambda'$, then

$$H_{D_n}(\lambda\lambda', t) = \frac{1}{2} H_{D_n}^-(\lambda\lambda', t), \quad P_{D_n}(\lambda\lambda', t) = \frac{1}{2} P_{D_n}^-(\lambda\lambda', t).$$

Proof. This follows from Proposition 3.8 and Lemma 3.11. \square

7. THE SPIN FAKE DEGREES OF TYPE D_n (FOR n EVEN)

Let n be even throughout this section.

7.1. The algebra $\mathbb{C}D_n^-$ for n even. Let \mathcal{SOP} be the set of strict odd partitions, i.e. those odd partitions containing no repeated parts.

Lemma 7.1. (cf. [Re2]) *Let n be even.*

- (1) *The split classes of $\mathbb{C}D_n^-$ are the classes of cycle types (ρ_+, ρ_-) in $(\mathcal{OP}, \mathcal{EP})$ or in $(\emptyset, \mathcal{SOP})$, with $\ell(\rho_-)$ even.*
- (2) *All split classes are even.*

Proof. Part (1) was proved in [Re2, Lemma 6.4]. Since all generators t_i are odd, the parity of an element $t_{i_1} \cdots t_{i_k}$, and thus of its conjugacy class, is equal to the parity of k . Now (2) can be extracted from [Re2]. \square

By [Re2, Lemma 8.1], the number of these split classes in D_n is equal to the number of conjugacy classes of the alternating group A_n . In the spirit of Theorem 6.2, the work of Read suggests the following structure result for $\mathbb{C}D_n^-$.

Conjecture 7.2. *For n even, we have a superalgebra isomorphism $\mathcal{C}l_n \otimes \mathbb{C}A_n \cong \mathbb{C}D_n^-$, where $\mathbb{C}A_n$ is even while the n generators of $\mathcal{C}l_n$ are odd.*

7.2. Simple $\mathbb{C}D_n^-$ -modules for n even. Recall the simple $\mathbb{C}B_n^-$ -modules B^λ from Proposition 4.3, which are all of type \mathbb{M} since n is even. Read [Re2, Lemma 8.4, Corollary 8.12] has classified the ungraded irreducible $\mathbb{C}D_n^-$ -modules. Recall from Proposition 4.3 the simple $\mathbb{C}B_n^-$ -modules B^λ , for $\lambda \vdash n$, and set

$$(7.1) \quad D^{\{\lambda, \lambda'\}} := B^\lambda|_{\mathbb{C}D_n^-}, \quad (\text{for } n \text{ even}).$$

According to Read, $|D^{\{\lambda, \lambda'\}}|$ is a simple $|\mathbb{C}D_n^-|$ -module if $\lambda \neq \lambda'$. In the case when $\lambda = \lambda'$, $D^{\{\lambda, \lambda'\}}$ is a sum of two inequivalent simple $|\mathbb{C}D_n^-|$ -modules D_{\pm}^{λ} :

$$(7.2) \quad D^{\{\lambda, \lambda'\}} = D_+^{\lambda} \oplus D_-^{\lambda}.$$

All these simple $|\mathbb{C}D_n^-|$ -modules can be endowed with the structures of simple $\mathbb{C}D_n^-$ -modules by Remark 2.4, since the graded irreducibles are all of type \mathbb{M} , by Proposition 2.5 and Lemma 7.1. Hence, Read's results can be upgraded as follows.

Proposition 7.3. *Let n be even. A complete list of pairwise inequivalent simple $\mathbb{C}D_n^-$ -modules consists of $D^{\{\lambda, \lambda'\}}$ when $\lambda \neq \lambda'$ and D_{\pm}^{λ} when $\lambda = \lambda'$, for $\lambda \vdash n$. All these simple $\mathbb{C}D_n^-$ -modules are of type \mathbb{M} .*

It is well known that the simple A_n -modules are parametrized in the same way as the simple $\mathbb{C}D_n^-$ -modules above. Read's classification of the simple $|\mathbb{C}D_n^-|$ -modules can be reformulated as stating that the ungraded version of the isomorphism in Conjecture 7.2 holds. On the other hand, Conjecture 7.2, if established by a direct and constructive proof, would immediately provide a new proof of the classification of simple $\mathbb{C}D_n^-$ -modules.

7.3. Spin fake degrees of D_n for n even. We continue the notation $H_{D_n}^-(\lambda\lambda', t)$ and $P_{D_n}^-(\lambda\lambda', t)$ from (6.5) when $\lambda \neq \lambda'$, now for n even. In addition, we will write $H_{D_n}^-(\lambda_{\pm}, t)$ and $P_{D_n}^-(\lambda_{\pm}, t)$ to indicate the graded multiplicities of the simple modules D_{\pm}^{λ} in $\mathcal{B}_{D_n} \otimes S^*V$ and $\mathcal{B}_{D_n} \otimes (S^*V)_{D_n}$, when $\lambda = \lambda'$, for n even.

Theorem 7.4. *Let n be even. Then we have*

$$\begin{cases} H_{D_n}^-(\lambda_+, t) = H_{D_n}^-(\lambda_-, t) = t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1 + t^{2c_{\square} + 1}}{1 - t^{2h_{\square}}}, & \text{if } \lambda = \lambda', \\ H_{D_n}^-(\lambda\lambda', t) = \frac{t^{2n(\lambda)}}{\prod_{\square \in \lambda} (1 - t^{2h_{\square}})} \left(\prod_{\square \in \lambda} (1 + t^{2c_{\square} + 1}) + \prod_{\square \in \lambda} (t^{2c_{\square}} + t) \right), & \text{if } \lambda \neq \lambda'. \end{cases}$$

Proof. As in the proof of Theorem 6.12, we write

$$\mathcal{B}_{D_n} \otimes S^*V \cong (\mathcal{B}_{B_n} \otimes S^*V)|_{\mathbb{C}D_n^-} = \bigoplus_{\lambda \vdash n} H_{B_n}^-(\lambda, t) B^{\lambda}|_{\mathbb{C}D_n^-},$$

since all B^{λ} are type \mathbb{M} when n is even. On the other hand, it follows by definition that

$$\mathcal{B}_{D_n} \otimes S^*V = \bigoplus_{\{\lambda \neq \lambda'\}} H_{D_n}^-(\lambda\lambda', t) D^{\{\lambda, \lambda'\}} \oplus \bigoplus_{\lambda = \lambda'} (H_{D_n}^-(\lambda_+, t) D_+^{\lambda} \oplus H_{D_n}^-(\lambda_-, t) D_-^{\lambda}).$$

A comparison of the above two identities using (7.1) and (7.2) gives us

$$\begin{cases} H_{D_n}^-(\lambda\lambda', t) = H_{B_n}^-(\lambda, t) + H_{B_n}^-(\lambda', t), & \text{if } \lambda \neq \lambda', \\ H_{D_n}^-(\lambda_{\pm}, t) = H_{B_n}^-(\lambda, t), & \text{if } \lambda = \lambda'. \end{cases}$$

Now the theorem follows from these two identities, the formula for $H_{B_n}^-(\lambda, t)$ in Theorem 4.9, and the identity (6.7) with $p(n) = 0$. \square

The following is equivalent to Theorem 7.4 by Lemma 3.11 and using the fact that the degrees of D_n are $2, 4, \dots, 2n-2, n$.

Theorem 7.5. *Let n be even. Then the spin fake degree $P_{D_n}^-(\lambda\lambda', t)$ is*

$$\begin{cases} P_{D_n}^-(\lambda_\pm, t) = t^{2n(\lambda)} \prod_{\square \in \lambda} \frac{1+t^{2c_\square+1}}{1-t^{2h_\square}} \prod_{i=1}^{n-1} (1-t^{2i})(1-t^n), & \text{if } \lambda = \lambda', \\ P_{D_n}^-(\lambda\lambda', t) = \frac{t^{2n(\lambda)}}{\prod_{\square \in \lambda} (1-t^{2h_\square})} \left(\prod_{\square \in \lambda} (1+t^{2c_\square+1}) + \prod_{\square \in \lambda} (t^{2c_\square} + t) \right) \prod_{i=1}^{n-1} (1-t^{2i})(1-t^n), & \text{if } \lambda \neq \lambda'. \end{cases}$$

Proposition 7.6. *The following palindromicity holds for spin fake degrees of D_n : for each irreducible $\mathbb{C}D_n^-$ -character χ , we have $P_{D_n}^-(\chi, t) = t^{n(n-1)} P_{D_n}^-(\chi, t^{-1})$.*

Proof. The proof is similar to that of Proposition 6.14. \square

7.4. Hecke-Clifford algebra $\mathfrak{H}_{D_n}^\epsilon$ for n even. Recall from Proposition 5.4 that the simple $\mathfrak{H}_{B_n}^\epsilon$ -modules K^λ are parametrized by $\lambda \vdash n$ and are all of type M.

Proposition 7.7. *Let n be even.*

- (1) *If $\lambda \neq \lambda'$, then $K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}$ is a simple $\mathfrak{H}_{D_n}^\epsilon$ -module, and $K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon} \cong K^{\lambda'}|_{\mathfrak{H}_{D_n}^\epsilon}$.*
- (2) *If $\lambda = \lambda'$, then $K^\lambda|_{\mathfrak{H}_{D_n}^\epsilon}$ is a sum of two inequivalent simple $\mathfrak{H}_{D_n}^\epsilon$ -modules. All these simple modules are of type M.*
- (3) *The simple modules in (1) and (2) (modulo the identifications in (1)) form a complete list of inequivalent simple $\mathfrak{H}_{D_n}^\epsilon$ -modules.*

Proof. By Proposition 3.1, there is a Morita super-equivalence between $\mathbb{C}D_n^-$ and $\mathfrak{H}_{D_n}^\epsilon$, and hence Proposition 3.3 applies. By Lemma 5.3, K^λ corresponds to B^λ under the super-equivalence. Now the proposition follows. \square

The list of simple $\mathfrak{H}_{D_n}^\epsilon$ -modules in Proposition 7.7 corresponds bijectively via the Morita super-equivalence to the list of simple $\mathbb{C}D_n^-$ -modules in Proposition 7.3. Proposition 3.8 ensures that the graded multiplicities of a simple $\mathfrak{H}_{D_n}^\epsilon$ -module in $\mathcal{C}l_V \otimes S^*V$ (respectively, in $\mathcal{C}l_V \otimes (S^*V)_{D_n}$) are exactly the same as their counterparts in $\mathcal{B}_{D_n} \otimes S^*V$ (respectively, in $\mathcal{B}_{D_n} \otimes (S^*V)_{D_n}$) given in Section 7.3.

8. THE SPIN FAKE DEGREES OF EXCEPTIONAL WEYL GROUPS

Let W be an exceptional Weyl group throughout this section.

8.1. List of split classes. We first briefly recall Carter's parametrization of conjugacy classes of Weyl groups by admissible diagrams [Ca] as follows. Given a conjugacy class, choose a representative element w , and decompose it into a product of two involutions subject to certain conditions (see [Ca, Section 3]). Each involution is a product of reflections corresponding to mutually orthogonal roots, and the admissible diagram is a graph whose nodes correspond to (the roots associated to) the reflections, with the edge between nodes corresponding to roots r and s having weight $4 \frac{(r,s)(s,r)}{(r,r)(s,s)}$. Many of the

possible graphs resemble Dynkin diagrams and are named accordingly. Carter shows that if w has an admissible diagram Γ , then so must all its conjugates [Ca, p.6], and that we may describe the conjugacy classes of any Weyl group W by such admissible diagrams [Ca, pp.45].

Morris [Mo1] has determined the split conjugacy classes for the exceptional Weyl groups W (with the double covers \widetilde{W}), and their descriptions are given in terms of Carter's parametrization by admissible diagrams.

We also need to determine the parity of the split classes, and this can be done in a simple and precise manner. It turns out that the nodes of the admissible diagram that labels a conjugacy class of W correspond to the reflections in a certain decomposition of an element in that class. Since all reflections in W are odd, the parity of each split conjugacy class of W may be read off from the number of nodes in the corresponding admissible diagram. Below we summarize Morris' classification of split classes, enhanced by the parity separation.

Proposition 8.1. *A complete list of split classes of every exceptional Weyl group W is as follows.*

- (E_6) *There are 9 even split classes: $\emptyset, A_2, A_4, 2A_2, D_4(a_1), 3A_2, E_6, E_6(a_1), E_6(a_2)$. There are 4 odd split classes: $D_5, D_5(a_1), D_3 + D_2$, and $A_4 + A_1$.*
- (E_7) *There are 13 even split classes: $\emptyset, D_4(a_1), A_2, 2A_2, 3A_2, E_6(a_2), E_6, E_6(a_1), A_4, A_4 + A_2, D_6(a_1), A_6$, and D_6 ; There are 13 odd split classes: $7A_1, 2A_3 + A_1, D_4 + 3A_1, D_6(a_2) + A_1, E_7(a_4), A_5 + A_2, E_7(a_2), E_7, D_6 + A_1, E_7(a_3), A_7, E_7(a_1), A_4 + A_1$.*
- (GFE) *All 3, 9, 30 split classes of G_2, F_4 , and E_8 listed in [Mo1, §8, §9] are even.*

8.2. Classification of simple modules. Now Proposition 2.5 and Lemma 8.1 allow us to determine the numbers of simple $\mathbb{C}W^-$ -modules of type \mathbf{M} and type \mathbf{Q} . Morris [Mo1] (also cf. Read [Re1] for F_4) has determined the spin character table of all the *ungraded* simple characters of $|\mathbb{C}W^-|$. It turns out that we may determine which pairs of ungraded simple characters add to become type \mathbf{Q} characters of $\mathbb{C}W^-$ since the character values for any $\mathbb{C}W^-$ -module on the odd split classes must be 0.

We will follow Morris's notation [Mo1], referring to the irreducible spin characters for the exceptional Weyl groups by their degrees with a subscript s (or $ss, sss, ssss$, for multiple characters of the same degree). In cases where two ungraded simple characters of the same degree add to become a type \mathbf{Q} character (which only happens in E_6, E_7), we will denote the type \mathbf{Q} character by its degree with the shortest possible subscript and a superscript \mathbf{Q} .

The Weyl group E_7 is the direct product of the Chevalley group $B_3(2)$ and a cyclic group $\langle \xi \rangle$ of order 2, where all elements in $B_3(2)$ are even while the generator ξ is odd. Hence the simple E_7 -modules are exactly the tensor products of simple $B_3(2)$ -modules with the unique two-dimensional simple module of $\langle \xi \rangle$ (of type \mathbf{Q}), and they are manifestly all of type \mathbf{Q} . Table E summarizes the conversion between the new type \mathbf{Q} notation and Morris's ungraded simple characters.

Table E: Type \mathbf{Q} characters as sums of ungraded simple characters

E_6	Type Q character	120_{ss}^Q	160_s^Q	40_{sss}^Q	128_s^Q	
	Ungraded characters	$60_s + 60_{ss}$	$80_s + 80_{ss}$	$20_s + 20_{ss}$	$64_s + 64_{ss}$	
E_7	Type Q character	16_s^Q	96_s^Q	336_s^Q	560_s^Q	224_s^Q
	Ungraded characters	$8_s + 8_{ss}$	$48_s + 48_{ss}$	$168_s + 168_{ss}$	$280_s + 280_{ss}$	$112_s + 112_{sss}$
	Type Q character	224_{ss}^Q	1024_s^Q	1440_s^Q	1120_s^Q	896_s^Q
	Ungraded characters	$112_{ss} + 112_{ssss}$	$512_s + 512_{ss}$	$720_s + 720_{ss}$	$560_s + 560_{ss}$	$448_s + 448_{ss}$
	Type Q character	240_s^Q	128_s^Q	128_{ss}^Q		
	Ungraded characters	$120_s + 120_{ss}$	$64_s + 64_{sss}$	$64_{ss} + 64_{ssss}$		

In this way, we have upgraded the results of Morris into the following.

Proposition 8.2. *The types of the simple $\mathbb{C}W^-$ modules, for W an exceptional Weyl group, are as follows.*

- (1) $\mathbb{C}E_6^-$ has 5 type M simple modules $8_s, 40_s, 72_s, 40_{ss}, 120_s$, and 4 type Q simple modules $120_{ss}^Q, 160_s^Q, 40_{sss}^Q, 128_s^Q$.
- (2) All 13 simple $\mathbb{C}E_7^-$ -modules are of type Q .
- (3) All 30 simple $\mathbb{C}E_8^-$ -modules are of type M , as are all 9 simple $\mathbb{C}F_4^-$ -modules and all 3 simple $\mathbb{C}G_2^-$ -modules.

8.3. Spin fake degrees. The main computational tool for the spin fake degrees of the exceptional Weyl groups is the spin Molien's formula, see Proposition 3.13 or Corollary 3.14. We will implement this using CHEVIE; see code in Section 8.4. To that end, as inputs we shall need the spin character tables of $\mathbb{C}W^-$, which were computed by Morris [Mo1] in the ungraded setting.

More precisely, we use the spin character tables of Morris for E_6 [Mo1, Table III], E_7 [Mo1, Table IV], E_8 [Mo1, Table V], and G_2 [Mo1, Table VI]. In cases of E_6 and E_7 , we need to add suitable pairs of columns in the spin character tables to form the characters of type Q simple modules.

Remark 8.3. There is a typo in the E_8 spin character table: the thirteenth entry of the last character should be 2 rather than -2 , which is detected and corrected using the orthogonality relations of simple characters.

For W of type F_4 , Read has labeled its 9 simple characters as ϕ_1, \dots, ϕ_9 . See Table F for a comparison between Read's notation [Re1, Table 1] and Morris's [Mo1, Table VII].

Table F: Two labelings of the spin character table for F_4

Morris's labels	4_s	4_{ss}	8_{sss}	8_{ssss}	12_{ss}	12_s	8_s	24_s	8_{ss}
Read's labels	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9

The character values which we use in the computation of the spin fake degrees for F_4 are those given by Read [Re1, Table 1]. They differ by a sign in the conjugacy classes A_2, \tilde{A}_2 , and B_2 from those given by Morris [Mo1, Table VII] because of a different choice of these \tilde{F}_4 -conjugacy classes (differing by a factor $z = -1$). However this does not affect the computation of the spin fake degrees.

We summarize our CHEVIE computations in Proposition 8.4 and Theorem 8.5 below. Recall the values for N , the number of reflections in W , given in Table D.

Proposition 8.4. *Let W be an arbitrary exceptional Weyl group. Then for every simple $\mathbb{C}W^-$ -character χ , we have*

$$P_W^-(\chi, t) = t^N P_W^-(\chi, t^{-1}).$$

Proof. Follows by inspection case-by-case from our CHEVIE computation. \square

We exploit this property when presenting the spin fake degrees for the exceptional groups in Table 1 through Table 5 in Section 9. In each column of the tables, we list only the coefficients of the spin fake degrees $P_W^-(\chi, t)$ (which are polynomials in t) for degrees 0 through $\frac{N}{2}$. The remaining halves of coefficients can be determined via palindromicity in Proposition 8.4.

Theorem 8.5. *For W exceptional, the coefficients of the spin fake degrees $P_W^-(\chi, t)$ are given in Table 1 through Table 5 in Section 9.*

Proof. These values are computed using Corollary 3.14. The computations are performed using the GAP 3 package CHEVIE [GAP, CHE], and the code is given below in Section 8.4. \square

The spin fake degrees for \mathfrak{H}_W^c may be obtained via Proposition 3.8, and hence are skipped.

8.4. Code. The following GAP function computes the spin fake degrees for the exceptional groups by implementing the formula given in Corollary 3.14. It requires a file of spin character tables to have been loaded; this file formats the spin character tables for the exceptional groups as regular GAP character tables (in particular one can use the DisplayCharTable function), although not all of GAP's character table functions will behave as desired with the spin tables.

```
# a function to compute spin fake degrees.
# -assumes that basic spin is the first character in the spin
#   table
# -takes t as a parameter
# -only supports exceptional groups (needs the corresponding
#   spin tables to have been loaded)

#-----
# SpinFakeDegrees(W,t )
# for W an exceptional Weyl group
# computes the 'graded multiplicities' of the W^- irreps in
# B tensor SV_W.
# returns a list of these multiplicities evaluated at t
#-----

SpinFakeDegrees := function( W,t )
  local p,q, ev, termmult, spinct, todet, ans;

  q:=X(Cyclotomics);
```

We set q to be an indeterminate, and at the end evaluate our answer at $q = t$. This allows us to use the same function to compute the spin fake degrees and their specializations to any value of t .

```

p:= GenericOrder(W,q); p.valuation:=0;
# p is the product of (q^d -1) over the degrees d
# this causes sign issues for E7 (expects product (1-q^d))
# which are fixed in the if statement later

#-----
# termmult(list1, list2)
# returns the list whose ith elt is the product of the
# ith elts of list1 and list2
#-----
termmult:= function( list1, list2)
    return List([1..Length(list1)], i->list1[i]*list2[i]); end;

ev:= ReflectionEigenvalues(W);
# this is a list of eigenvalues of the reflection representation
# by conjugacy class, but we only care
# about the even split conjugacy classes. The following giant if
# statement selects the appropriate classes worth of values for
# each group, and makes sure we're looking at the right spin table.

if W = CoxeterGroup("G",2)
    then ev:= List([1,5,4], i-> ev[i]); spinct := spinG2;
elif W = CoxeterGroup("F",4)
    then ev:= List([1,4,6,7,9,10,11,23,25], i->ev[i]); spinct := spinF4;
elif W = CoxeterGroup("E",6)
    then ev:= List([1,5,9,6,7,4,15,14,10], i->ev[i]); spinct := spinE6;
elif W = CoxeterGroup("E",7)
    then ev:= List([1,9,6,8,7,17,29,25,14,30,24,22,26], i->ev[i]);
    spinct := spinE7; p := -p;
elif W = CoxeterGroup("E",8)
    then ev:= List([1,8,14,16,62,41,44,53,57,46,33,49,64,26,56,11,
    23,12,48,25,10,55,65,39,
    21,6,29,61,52,32], i->ev[i]); spinct := spinE8;
else Print("Group not supported."); return false;
fi;

# Now to actually compute the spin fake degrees.

#-----
# todet(list)
# takes a list of lists of eigenvalues lambda as returned by

```

```

# ReflectionEigenvalues, reformats them and returns the product of
# 1 - lambda*q for each list
#-----
todet:= x -> Product(x, n -> 1-(E(Denominator(n))^Numerator(n))*q);

ans := List(ev, todet); # the list of determinants (det (1-phi q))
ans := termmult(spinct.centralizers, ans); # the list of determinants
# times centralizer orders
ans := List(ans, x->p/x);
ans := termmult(ans, spinct.irreducibles[1]); # multiply by basic spin
ans := spinct.irreducibles*ans;

return List(ans, x->Value(x,t));
end;

```

9. TABLES FOR SPIN FAKE DEGREES OF EXCEPTIONAL TYPES

We use the notation and convention as specified in Section 8. In particular, the characters with superscripts \mathbf{Q} in Table 1 through Table 5 are of type \mathbf{Q} , and those without are of type \mathbf{M} . Note that in all tables, the character of the basic spin module \mathcal{B}_W is listed first.

TABLE 2. Spin fake degrees for type F_4 TABLE 1. Spin fake degrees for type G_2

	2_s	2_{ss}	2_{sss}
0	1		
1	1		1
2	0	1	1
3	0	2	0

	4_s	4_{ss}	8_{sss}	8_{ssss}	12_{ss}	12_s	8_s	24_s	8_{ss}
0	1								
1	1						1		
2	0						1	1	
3	0		1	1			0	2	
4	0		2	2			1	2	1
5	1		1	1	1	2	1	3	1
6	1	1	0	0	3	3	2	4	0
7	1	2	1	1	4	3	2	4	2
8	1	1	3	3	3	2	3	5	3
9	0	0	3	3	2	2	2	8	2
10	0	0	2	2	4	3	0	9	2
11	1	2	2	2	5	4	3	7	3
12	2	4	2	2	4	4	6	6	4

TABLE 3. Spin fake degrees for type E_6

	8_s	40_s	72_s	40_{ss}	120_s	$120_s^{\mathbf{Q}}$	$160_s^{\mathbf{Q}}$	$40_{sss}^{\mathbf{Q}}$	$128_s^{\mathbf{Q}}$
0	1								
1	1	1							
2	0	1				2			
3	0	1			1	4			
4	1	3	1		2	4	2		
5	2	4	3		3	6	4		2
6	1	4	4		7	10	6		4
7	1	5	5	1	11	14	12		6
8	2	8	9	3	13	18	18	4	12
9	2	9	13	6	18	24	24	8	18
10	1	9	14	8	27	30	34	6	24
11	2	11	19	8	34	34	44	8	34
12	4	14	27	13	39	38	52	14	44
13	3	15	29	19	47	44	64	16	50
14	1	14	30	18	55	52	74	18	60
15	2	16	35	20	59	56	80	24	70
16	4	19	39	28	62	56	88	26	72
17	3	18	40	26	67	58	92	24	76
18	2	16	40	20	70	60	92	24	80

TABLE 4. Spin fake degrees for type E_7

	16_s^Q	96_s^Q	336_s^Q	560_s^Q	224_s^Q	224_{ss}^Q	1024_s^Q	1440_s^Q	1120_s^Q	896_s^Q	240_s^Q	128_s^Q	128_{ss}^Q
0	2												
1	2	2											
2	0	2	2										
3	0	0	4	2									
4	0	2	4	4			2						
5	2	4	6	4	2		4	2					
6	2	6	8	6	4		6	6			2		
7	2	6	10	10	4		12	10		2	4		
8	2	6	14	16	6		18	16	4	4	2		
9	2	8	20	22	8	2	26	24	12	6	2		
10	2	10	26	28	10	4	38	36	18	14	6		
11	2	12	30	36	14	4	52	56	26	22	10	2	2
12	4	14	36	46	20	8	68	80	40	32	14	4	4
13	4	18	44	58	26	14	90	104	60	50	18	4	4
14	4	18	54	72	30	18	116	136	86	68	22	8	8
15	2	18	64	90	34	22	142	178	116	90	28	12	12
16	4	22	72	108	40	30	176	222	150	122	36	14	14
17	6	28	82	124	52	40	212	274	192	154	44	20	20
18	6	30	94	142	62	48	250	334	240	188	56	26	26
19	4	30	104	166	66	58	294	396	288	234	68	30	30
20	4	32	116	190	74	68	338	462	346	278	76	38	38
21	6	36	130	212	86	82	382	530	410	322	86	48	48
22	6	40	140	234	94	96	430	600	468	376	100	52	52
23	6	40	148	256	102	104	476	676	526	424	114	60	60
24	6	42	158	278	112	116	518	748	588	468	124	70	70
25	8	46	170	298	120	132	562	806	648	520	134	74	74
26	6	48	180	316	126	142	600	866	702	562	144	82	82
27	6	46	186	332	130	146	632	926	748	594	154	90	90
28	6	48	190	346	136	156	662	968	784	632	164	92	92
29	8	52	196	354	144	166	684	1000	818	656	168	96	96
30	8	52	200	362	148	168	696	1026	842	668	170	102	102
31	6	50	200	368	142	168	706	1038	848	682	174	100	100

TABLE 5. Spin fake degrees for type E_8 (Part 1)

	16_s	112_s	320_s	448_s	224_s	448_{ss}	1680_s	2592_s	1344_s	5600_s	4800_s	2016_s	5600_{ss}	9072_s	800_s
0	1														
1	1	1													
2	0	1				1									
3	0	0				1									
4	0	0				0						1			
5	0	0				1						1	1		
6	0	1				2	1					0	2		
7	1	2	1			2	2					1	2		1
8	1	2	2			2	2					2	3		2
9	0	1	1			3	2					3	5	1	1
10	0	1	0			3	2					5	8	2	0
11	1	2	1			3	3					7	11	3	1
12	1	3	3			5	6	1		1		7	13	6	3
13	1	4	4	1		7	10	3		4	1	7	17	8	4
14	1	4	4	2		8	11	5		6	3	10	24	10	4
15	0	2	3	1		8	11	5		6	3	15	31	18	4
16	0	1	2	0		8	13	4		7	3	21	39	29	5
17	1	4	4	1		10	16	8		11	8	25	50	35	6
18	2	7	8	4		14	23	14	2	21	14	25	61	42	8
19	2	8	11	6	1	16	32	19	5	32	17	28	71	57	13
20	2	7	12	6	2	16	35	24	6	38	22	38	86	75	16
21	1	5	9	6	1	17	35	26	7	45	32	49	109	97	14
22	0	4	6	5	0	20	40	30	7	56	44	59	133	126	13
23	1	7	11	6	1	23	50	41	9	71	56	67	154	156	20
24	3	11	20	12	4	27	65	55	18	98	69	71	175	186	30
25	2	12	23	18	5	32	80	71	26	132	89	76	203	222	34
26	1	10	20	19	4	34	86	85	28	155	116	92	241	267	34
27	1	8	18	16	5	33	86	90	32	171	137	117	281	328	36
28	1	8	18	15	6	35	95	98	39	197	159	138	320	398	41
29	2	12	24	21	7	42	115	125	46	243	200	146	364	456	49
30	4	18	36	34	10	49	139	159	63	308	244	150	408	512	60
31	4	19	42	43	16	52	161	183	84	369	277	165	452	593	72
32	2	14	38	39	19	53	170	203	90	407	319	194	510	694	78
33	1	11	31	35	14	55	171	220	93	445	377	228	580	799	75
34	1	13	31	39	12	60	185	241	107	506	436	253	646	906	78
35	2	17	43	48	20	68	217	284	126	584	493	265	702	1013	98
36	4	23	59	64	29	74	253	335	157	682	552	274	758	1118	120
37	5	25	64	78	34	78	277	373	187	782	622	294	829	1231	128
38	2	20	55	75	34	81	283	401	193	843	705	332	915	1369	127
39	1	15	48	65	31	81	285	421	200	889	776	379	999	1531	131
40	3	18	54	70	35	83	305	448	227	973	842	411	1073	1684	145
41	4	25	68	90	43	94	346	511	257	1094	936	417	1145	1804	163
42	4	30	82	111	50	105	389	584	291	1229	1031	419	1218	1924	182
43	5	30	87	120	58	106	413	622	328	1336	1100	449	1292	2080	197
44	3	24	78	110	61	104	413	641	337	1387	1180	501	1380	2257	200
45	0	18	65	99	52	108	413	667	336	1442	1281	545	1477	2426	196
46	1	22	69	111	48	114	437	706	367	1551	1370	567	1556	2571	204
47	5	31	92	135	69	120	482	770	410	1681	1447	571	1613	2692	233
48	6	35	110	154	87	127	525	838	448	1807	1523	574	1670	2814	261
49	5	34	107	161	83	130	540	873	476	1906	1603	600	1745	2951	263
50	4	29	92	150	78	128	527	881	473	1940	1692	649	1832	3098	252
51	2	23	82	135	74	127	522	891	471	1962	1760	693	1903	3251	254
52	2	25	90	143	75	131	549	925	504	2052	1807	706	1948	3372	271
53	5	35	111	173	92	140	592	989	542	2179	1878	690	1987	3428	291
54	6	40	124	193	104	148	624	1047	566	2278	1947	680	2027	3482	304
55	4	35	118	186	99	147	627	1053	580	2316	1974	706	2066	3589	308
56	3	28	102	165	93	140	603	1030	569	2290	2004	753	2112	3694	301
57	2	25	91	155	87	139	587	1028	553	2280	2054	779	2155	3753	289
58	2	28	97	169	84	145	609	1055	575	2347	2078	767	2169	3775	294
59	5	37	121	193	104	149	645	1095	608	2420	2079	741	2156	3766	319
60	8	42	136	204	122	150	660	1116	618	2442	2080	730	2146	3754	334

TABLE 5. Spin fake degrees for type E_8 (Part 2)

	2800 _{ss}	5600 _{sss}	7168 _s	1120 _s	8400 _s	11200 _s	6720 _s	2800 _s	1344 _{ss}	6480 _s	8192 _s	2016 _{ss}	2016 _{sss}	7168 _{ss}	896 _s
0															
1															
2															
3									1						
4									2						
5									2						
6		1							2						
7		2					1	1	2						
8		2	1				2	2	3	1					
9		3	3			1	2	1	5	1					
10		4	5		1	3	3	0	7	0					
11	1	5	7		2	4	5	2	8	2	1				
12	2	9	8		2	5	8	6	9	5	2			1	
13	1	15	9		3	9	12	8	10	7	3			2	
14	1	19	14		5	15	16	9	12	9	6			3	
15	4	21	23		10	20	20	10	18	13	9			6	
16	7	24	32		18	27	26	11	24	18	13			9	1
17	8	33	41		24	39	35	15	25	23	20	1	1	13	1
18	9	49	49	1	28	53	47	23	25	31	28	3	3	20	0
19	12	64	57	4	36	68	62	34	29	45	37	6	6	28	2
20	18	74	75	7	51	87	77	42	36	60	51	10	9	37	4
21	25	84	103	6	72	114	91	41	45	68	68	10	10	51	4
22	31	99	129	3	98	147	109	41	54	79	86	10	12	68	5
23	40	121	151	8	122	180	135	58	59	106	111	17	17	86	9
24	50	154	173	18	142	217	168	84	61	140	140	26	26	111	12
25	56	192	198	21	169	269	203	101	67	168	171	35	35	140	14
26	67	219	240	22	212	331	235	109	79	196	212	43	40	170	18
27	91	237	302	27	271	390	271	116	95	229	258	49	49	210	22
28	115	266	360	32	335	457	316	129	109	268	306	58	63	256	29
29	128	318	403	39	388	544	368	156	114	317	366	74	74	302	35
30	141	386	446	52	434	638	430	195	115	375	432	94	91	360	38
31	166	445	501	68	499	732	497	232	126	442	501	116	116	425	48
32	200	483	579	80	594	838	557	252	148	508	583	136	131	490	61
33	235	521	681	79	703	962	616	255	169	557	672	145	145	568	66
34	266	579	775	80	810	1095	690	270	183	614	762	159	169	655	73
35	298	659	840	106	902	1228	778	326	191	713	866	196	196	739	90
36	333	754	902	138	984	1367	874	397	196	824	978	236	230	835	104
37	364	841	991	153	1090	1528	970	435	207	907	1088	265	265	943	112
38	405	897	1113	159	1236	1702	1052	445	232	981	1212	291	286	1045	126
39	467	939	1252	169	1401	1863	1135	459	261	1069	1343	311	311	1158	142
40	522	1011	1371	187	1548	2027	1239	496	277	1168	1469	338	353	1283	158
41	550	1128	1448	214	1658	2219	1351	562	279	1283	1608	389	389	1397	175
42	579	1253	1516	244	1758	2413	1464	637	282	1407	1752	438	425	1520	186
43	634	1339	1625	270	1898	2589	1576	687	299	1521	1887	473	473	1656	203
44	700	1383	1776	286	2085	2771	1669	700	330	1617	2032	507	504	1777	227
45	756	1431	1929	284	2270	2968	1752	698	358	1695	2179	524	524	1902	240
46	799	1521	2042	291	2413	3159	1858	731	368	1786	2314	547	566	2038	249
47	837	1641	2106	341	2515	3335	1980	822	366	1926	2455	614	614	2156	273
48	876	1755	2163	390	2610	3505	2092	912	369	2068	2596	674	655	2273	294
49	917	1831	2269	393	2744	3681	2185	934	384	2147	2719	693	693	2402	300
50	967	1858	2421	388	2921	3855	2255	913	410	2197	2844	710	713	2508	314
51	1028	1878	2554	397	3087	3998	2320	915	437	2274	2967	731	731	2605	334
52	1071	1949	2621	415	3187	4123	2406	963	445	2369	3067	753	772	2714	346
53	1076	2064	2639	450	3228	4264	2497	1038	432	2464	3166	806	806	2798	356
54	1083	2157	2661	479	3272	4392	2569	1097	426	2547	3260	853	827	2868	364
55	1126	2177	2734	481	3369	4475	2619	1105	444	2599	3327	853	853	2949	372
56	1174	2149	2849	475	3499	4546	2644	1072	470	2620	3391	852	861	3002	386
57	1194	2146	2929	466	3586	4624	2661	1045	483	2626	3448	858	858	3036	392
58	1192	2198	2925	465	3592	4677	2696	1070	477	2654	3476	860	876	3081	387
59	1186	2265	2877	504	3553	4696	2738	1144	460	2718	3498	898	898	3100	394
60	1184	2294	2852	536	3530	4700	2756	1188	450	2756	3512	932	904	3096	404

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